

THE 28th ANNUAL (2006) UNIVERSITY OF MARYLAND
HIGH SCHOOL MATHEMATICS COMPETITION
PART I SOLUTIONS

1. Let d be the number of dimes. Then there are d nickels and $2d$ quarters. The total amount of money is $26.00 = .05d + .10d + .50d = .65d$. Therefore $d = 40$. The answer is **d**.
2. The diamond has vertices at $(\pm 4, 0)$ and $(0, \pm 6)$. Therefore, it can be broken into 4 triangles of area 12, so the total area is 48. Or, you can take $1/2$ the product of the diagonals to get 48. The answer is **c**.
3. Multiply the first equation by -4 to get $-12x - 8y = -32$. If $a = -12$, the second equation is $-12x - 8y = 9$, which is inconsistent with the first equation. Another way: The first equation yields $y = 4 - (3/2)x$. Substitute into the second equation to get $ax - 32 + 12x = 9$. This becomes $(a + 12)x = 41$. If $a = -12$, we get $0 = 41$, so there is no solution (and if $a \neq -12$, there is a solution). The answer is **e**.
4. Since $(2^{1/2})^6 = 8$ is smaller than $(5^{1/3})^6 = 25$, we have $5^{1/3}$ is bigger than $2^{1/2}$. Since $(5^{1/3})^{12} = 625 > (8^{1/4})^{12} = 512$, we have $5^{1/3}$ is larger than $8^{1/4}$. Since $(5^{1/3})^{15} = 3125 > (11^{1/5})^{15} = 1331$, and $(5^{1/3})^6 > (14^{1/6})^6$, we conclude that $5^{1/3}$ is the largest. The answer is **b**.
5. Let f be the number of four-legged dinosaurs and t be the number of two-legged dinosaurs. Then $100 = f + t$ and $260 = 4f + 2t$. Doubling the first equation and subtracting it from the second yields $60 = 2f$, so $f = 30$. The answer is **a**.
6. To find the intersection, set $x^2 = 2x^2 - 3x + 2$, which yields $0 = x^2 - 3x + 2 = (x - 1)(x - 2)$. Therefore, $x = 1$ or 2 and the points of intersection are $(1, 1)$ and $(2, 4)$. The line through these points has slope 3. The answer is **d**.
7. We have $\log_2(x^4) = 4\log_2(x) = \log_2(81)$, so $x^4 = 81$. Therefore $x = 3$ (since it must be positive to be in $\log_2(x)$). The answer is **a**.
8. By the Pythagorean theorem, AC has length 5. Since AD has length 12, the Pythagorean theorem yields that CD has length 13. The answer is **c**.

9. In one hour, Manet can paint $1/3$ of the house and Monet can paint $1/2$ of the house, so they paint $1/2 + 1/3 = 5/6$ of the house in 1 hour. Therefore, it takes $1.2 = 6/5$ hours to paint the house. The answer is **b**.
10. Draw a right triangle with legs of length H and 200 and an angle of 15° . Then $\tan 15^\circ = H/200$, so $H = 200 \tan 15^\circ$. The answer is **c**.
11. The sum of three consecutive terms is $2^j + 2^{j+1} + 2^{j+2} = 2^j(1 + 2 + 4) = 7 \cdot 2^j$, which is a multiple of 7. The sum $4 + 8 + 16 + \dots + 2^{1000}$ can be broken into 333 sums of consecutive powers of 2, hence is a multiple of 7. Therefore, what remains is $1 + 2$. Therefore, the remainder is 3. Another way to look at this: the remainders after dividing the powers of 2 by 7 cycle through 1, 2, 4. Therefore, each cycle sums to $0 \pmod{7}$. Therefore, the sum through 2^{998} is a multiple of 7. The remaining two terms yield remainders of 1 and 2, so the sum is $3 \pmod{7}$. The answer is **d**.
12. If n is a multiple of 30, then it is also a multiple of 10 and of 15. Since only two are right, Hilbert must be wrong. If n is a multiple of 10 and 12, then it is a multiple of 60, hence a multiple of 30, which is not possible. Similarly, 10 and 15 yield the impossible 30, and 10 and 18 yield the impossible 30. So 10 is out. Again, 12 and 15 yield 30, which is impossible, and so do 15 and 18. The only remaining possibility is 12 and 18. In fact, $n = 36$ satisfies all the criteria, so the pair 12, 18 is possible. The answer is **d**.
13. If we add $60 + 50 + 40 = 150$, then all the people who have had exactly two diseases get counted twice and the ones who had all three get counted three times. Subtract $30 + 15 + 10 = 55$ to get 95. Now, all the people who have had all three have been counted three times, and then subtracted three times, so we add in these 5 to get 100 people who have had at least one disease. There are $150 - 100 = 50$ remaining students. The answer is **e**.
14. After one hour, train B has traveled 30 miles and the fly has traveled 70 miles. Since they started 100 miles apart, the fly is on B. Train A has traveled 10 miles, so it is 60 miles away. Since the fly is going 7 times as fast as train A, it flies $7/8$ of 60 miles to reach train A. The total

distance flown by the fly is $70 + (7/8)60 = 122.5$ miles. (Additional note: when train A sees the fly return, it realizes that train B is approaching and stops. Therefore, the fly does not have to travel back and forth until it is crushed, and both the fly and the trains are saved.) The answer is **d**.

15. Note that $f(x, 1) = f(f(x, 0), 0) = f(x, 0) = x$. In fact, we claim that $f(x, y) = x$ for all nonnegative integers y : If $f(x, y) = x$, then $f(x, y + 1) = f(f(x, y), y) = f(x, y) = x$, so the claim follows by induction. Therefore, $f(14, 11) = 14$ is the largest. The answer is **e**.
16. Let m be the speed of the motor boat and r the speed of the river. Let d be the distance from A to B. Traveling downstream, the boat travels at a speed $m + r$ relative to the land, so $5(m + r) = d$. For upstream, we get $7(m - r) = d$. These yield $35r = d$. The time it takes the raft is $d/r = 35$ hours. The answer is **d**.
17. Let the 0 be in the m th position (start counting at the 1's place). Then n can be written as $10^m a + b$, with $b < 10^{m-1}$, and b has no 0's. Removing the 0 yields $r = 10^{m-1} + b$. Since $n = 9r$, we obtain $10^{m-1} a = 8b$. If $m \geq 5$, then $b = 10^{m-5}(1250)a$, which is not possible because b cannot have 0's. Therefore, $m \leq 4$. Since $b < 10^{m-1}$, we have $10^{m-1} a = 8b < 8 \times 10^{m-1}$, so $a < 8$. Therefore, a is a single digit. Since $m \leq 4$, there can be at most 5 digits in n . An example of such an n is 70875, and the above reasoning shows that it is the largest. The answer is **d**.
18. The graph of $\sin x$ is periodic with period 2π . Each period has a hill and a valley. Let's consider nonnegative x . The line $y = x/100$ intersects both sides of the hills until it gets above $y = 1$. This happens at $x = 100$. By this time, x has gone through $100/2\pi \approx 16$ periods. So we expect around 32 intersections for $x \geq 0$. Since we get the same number for $x \leq 0$, we get 63 intersections (since $x = 0$ was counted twice). This argument shows that the answer must be **d**. To make it more precise, we note that there are 30 intersections for $0 \leq x \leq 30\pi < 100$. Between 30π and 31π , we have a hill. Since $31\pi < 31 \times 3.15 < 100$, the line intersects the graph of $\sin x$ two more times in this range. We now have 32 intersections for $x \geq 0$. The next possible intersections are for $32\pi \leq x \leq 33\pi$. But $32\pi > 32 \times 3.14 > 100$, so the graph of $y = x/100$

is too high. This proves that there are exactly 32 intersections for $x \geq 0$. As before, this yields a total of 63 intersections. The answer is **d**.

19. A number differs from the sum of its digits by a multiple of 9. Therefore, each number eventually reduces to its remainder upon dividing by 9 (except that a multiple of 9 yields 9 instead of 0). Since 10^{2006} is 1 more than a number with 2006 nines, we get a list 1, 2, 3, ..., 9, 1, 2, 3, ..., 9, 1, 2, 3, ..., 9, 1. Therefore, there is one more 1 than 2's. The answer is **b**.
20. If the first player plays in the center, then the first player can play diagonally opposite the second player for the remainder of the game and thus get a draw. Therefore, I is false. If the first player plays in the corner and the second plays in the center, then the first player plays in the diagonally opposite corner. The second player then is forced to play in a way that he has two in a row and loses on the third turn. Therefore, II is true. If the first plays in a corner and the second in the opposite corner, then the first can play in the center, which reduces to the situation of I. Therefore, III is false. The answer is **e**.
21. Let the perpendicular from A to EF intersect EF at X and let the perpendicular from B to EF intersect EF at Y . Then EF is twice as long as XY . The segment XY is the projection of AB onto EF , so its largest possible length is that of AB , namely, 10. Therefore, the longest possibility for EF is 20. The answer is **b**.
22. To make x largest, we need the denominator smallest. The denominator is p plus a fraction y less than 1. Therefore, we must have $p = 1$. We now need y to be as small as possible. Since y has denominator q plus a fraction z less than 1, we need q as large as possible, so $q = 5$. We also need z as large as possible, so we take $r = 2$. Continuing in this way, we take $s = 4$ and $t = 3$. The answer is **c**.
23. The area of the triangle is $\frac{1}{2}ab \sin \theta$, where θ is the angle between the sides of lengths a and b . This is largest when $\theta = 90^\circ$ and the sides are $a = 1$ and $b = 2$. This is a right angle, so $c = \sqrt{5}$. Since $2 \leq \sqrt{5} \leq 3$, this triangle is allowed. Its area is 1. The answer is **a**.
24. We have

$$x = \frac{2-1}{2!} + \frac{3-1}{3!} + \cdots + \frac{1000-1}{1000!} = \frac{1}{1!} - \frac{1}{2!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{1}{999!} - \frac{1}{1000!}.$$

All of the terms cancel, except for the first and last, so we obtain $x = 1 - 1/1000!$. Since $1000!$ is a product of 1000 numbers, each of which is at most 1000, we have $1000! \leq 1000^{1000} = 10^{3000} < 10^{12345}$. Therefore, $.999 < x < 1 - 10^{-12345}$. The answer is **b**.

25. Subtracting the formula $\cos(x + y) = \cos x \cos y - \sin x \sin y$ from the same formula with $-y$ yields $\cos(x - y) - \cos(x + y) = 2 \sin x \sin y$. Therefore, $4 \sin(70) \sin(10) = 2 \cos(60) - 2 \cos(80) = 2(1/2) - 2 \sin(10) = 1 - 2 \sin(10)$. We find that

$$\frac{1}{\sin 10} - 2 \sin 70 = \frac{1 - 4 \sin(70) \sin(10)}{2 \sin 10} = \frac{2 \sin 10}{2 \sin 10} = 1.$$

The answer is **a**.