THE 30th ANNUAL (2008) UNIVERSITY OF MARYLAND

HIGH SCHOOL MATHEMATICS COMPETITION

PART I SOLUTIONS

1. Let m and ty be their ages today. Then $m+7=4(t+7)$ and $m+t=71$. Substituting $t=71-m$ into the first equation yields $m+7=4(78-m)$, which yields $m=61$. The answer is **(d)**.

2. Randy gets $4\times 5=20$ points for his correct answers but loses $2\times 20=40$ points for his wrong answers, thus receiving a score of -20. Bianca gets 12 points for correct answers but loses 6 points for wrong answers, thus receiving a score of 6. Randy's score minus Bianca's score is -26. The answer is **(e)**.

3. Let x be the height of the giraffe. Similar triangles shows that $5.5/4=x/11.2$. This yields $x=15.4$. The answer is **(e)**.

4. The equation has no real roots exactly when the discriminant (b^2-4ac in the quadratic formula) is negative. In this case, we have $4+4a<0$, which yields $a<-1$. The answer is **(a)**.

5. The best you can do is 3 quarters, 4 dimes, and 4 pennies. This is \$1.19. The answer is **(c)**.

6. The definition of logarithms says that $b^{\log_b(a)}=a$. Therefore, $2^{\log_2(3)}=3$. The expression becomes $3^{\log_3(4)}$, which equals 4. The answer is **(e)**.

7. Triangle ADB is a right triangle with hypotenuse 17 and leg 8. The other leg is $\sqrt{17^2-8^2}=15$. The area of ADB is 60. Traingle CBD is a right triangle with hypotenuse 10 and leg 8. The other leg is 6. The area of CBD is 24. The total area of the quadrilateral is $60+24=84$. The answer is **(b)**.

8. If you give 1 coin to the first friend, 2 coins to the second friend, etc., after 19 friends you have given away $1+2+\cdots+19=190$ coins. The 10 remaining coins are not enough for a 20th friend, so you give them to the 19th friend. In fact, 20 friends would require at least $1+2+\cdots+20=210$ coins, so 19 is the largest number of friends you could have. The answer is **(c)**.

9. Substitute $y=2-x$ into the second equation to obtain $x(2-x)-z^2=1$. This can be rewritten as $0=(x-1)^2+z^2$. Since x and z are real numbers, this implies that $x-1=z=0$, which yields the unique solution $x=1, y=1, z=0$. The answer is **(b)**.

10. Let d be the distance from A to B. The time from A to B is $d/40$. The time from B to A is $d/60$. The total time is therefore $(d/40)+(d/60)=d/24$. The average speed is distance divided by time, which is $(2d)/(d/24)=48$. (Why isn't the average 50? Average speed is an average with respect to time, and you spend a lot more time going at the slower speed). The answer is **(b)**.

11. Replacing any integer greater than or equal to 4 by a sum of 2's and 3's gives a larger product (for example, $2\times 2\times 3$ is larger than 7), so the product should be all 2's and 3's. Since $3\times 3>2\times 2\times 2$, there should not be more than two 2's. The largest product is $3\times 3\times 3\times 3\times 3\times 3\times 2=1458$. The answer is **(c)**.

12. Write $x=10a+b$, where $0\le a,b\le 9$. The sum of the digits of x is $a+b$, so we want $18=(10a+b)-(a+b)=9a$. This means $a=2$. There are 10 such numbers (20, 21, ..., 29). The answer is **(b)**.

13. Suppose it takes t seconds for Stu's stone to hit the ground. Then Stu's stone has fallen $16t^2$ feet, and Hal's stone has fallen $16(t-3)^2$ feet. Since Stu's stone falls 4 times as far as Hal's stone, $16t^2 = 4 \times 16(t-3)^2$. Divide by 16 and expand to obtain $0 = 3t^2 - 24t + 36 = 3(t-2)(t-6)$. Since $t = 2$ is before Hal drops his stone, we have $t = 6$. The distance Stu's stone fell is $16t^2 = 576$ feet. The answer is **(d)**.

14. Since $10^3 < 2^{10} < 2 \times 10^3$, we have $10^{30} < 2^{100} < 2^{10} \times 10^{30} < 10^{34}$. Therefore, the number of digits of 2^{100} is between 31 and 34. (In fact, it is 31.) The answer is **(d)**.

15. Let Bob's apartment number be $10n + m$, with $0 \le m \le 9$. Then $(10n + m) + n = 260$, which means $11n = 260 - m$. The only multiple of 11 of the form $260 - m$ with $0 \le m \le 9$ is 253, so $m = 7$ and $n = 23$. Bob's apartment number is 237 and Alice's is 23. The difference is 214. The answer is **(c)**.

16. Put all the angles into the first quadrant: $x = \sin 1, y = \sin(\pi - 2), z = \sin(\pi - 3)$. Since $0 < \pi - 3 < 1 < \pi - 2$, we have $z < x < y$. The answer is **(e)**.

17. If $2^{2^{k-1}} < n \le 2^{2^k}$, then Program I takes time k. If $2^{2\ell} \le < 2^{2(\ell+1)}$, then Program II takes time ℓ. We want $\ell < k$. Also, $2^{2^{k-1}} < n < 2^{2(\ell+1)}$, so $2^{k-1} < 2(\ell + 1)$, which implies that $2^{k-2} < \ell + 1 < k + 1$. Since both of these last two inequalities are not allowed to be equalities, we must have $2^{k-2} \le k - 1$. This occurs only for $k \le 3$. Since $\ell < k$, we have $\ell \le 2$, which means that $n < 2^{2(\ell+1)} \le 64$. In fact, $n = 63$ takes time $k = 3$ for I and time $\ell = 2$ for II, so 63 is the largest. The answer is **(a)**.

18. Let a be the largest element of a cool set X. If there is another element $b \ne a$ in X, then $a + b$ is larger than b and is in X by assumption. This contradicts the choice of b. Therefore, X can have only one element. There are 2008 one-element subsets of $\{1, \ldots, 2008\}$, and these are the only cool sets. The answer is **(c)**.

19. The polynomial $f(X) = P(X) - X - 2006$ satisfies $f(0) = f() = f(2) = 0$, so it has $X(X - 1)(X - 2)$ as a factor. It has degree 4, so there is one more factor, call it $X - b$. Then $P(X) = X + 2006 + X(X - 1)(X - 2)(X - b)$. The coefficient of X^3 is $-3 - b$, so b is an integer. We have $P(3) = 2009 + 6(3 - b) = 2027 - 6b$. The smallest positive value for this expression is when $b = 337$, which yields the value 5. The answer is **(b)**.

20. First, compute $S(1) + \cdots + S(999)$. Among the numbers from 1 to 999, each digit occurs in each position 1/10 of the time, so a given digit occurs 100 times in the units places, 100 times in the tens place, and 100 times in the hundreds place, for a total of 300 occurrences. Therefore, $S(1) + \cdots + S(999) = 300(0 + 1 + \cdots + 9) = 13500$. Now, $S(1000) + S(1001) + \cdots + S(1999)$ is the same sum except for one thousand 1's occurring as the first digits of the numbers. Therefore, $S(1000) + S(1001) + \cdots + S(1999) = 1000 + 13500 = 14500$. Finally, $S(2000) + S(2001) + \cdots + S(2008)$ has nine 2's plus $1 + 2 + \cdots + 8 = 36$, which adds up to 54. The overall total is $13500 + 14500 + 54 = 28054$. The answer is **(b)**.

21. If the first person gets the correct name tag, then all the others get wrong tags. Call these people B, C, D, E. The 9 possible orderings of these 4 tags are CBED, CDEB, CEBD, DBEC, DEBC, DECB, EBCD, EDBC, EDCB. Similarly, there are 9 possible orderings that yield each of the other people getting the correct tag, so there are 45 orderings that yield one person with a correct tag. There are 120 possible orderings, so the probability is 45/120=3/8. The answer is **(d)**.

22. Square both sides and subtract 1 to obtain $\sqrt{x} = x^2 - 2x = x(x-2)$. Square again to obtain $x = x^2(x-2)^2$. Since $x = 0$ is not a solution to the original equation, divide by x to obtain $1 = x(x-2)^2$, which rearranges to $x^3 - 4x^2 + 4x - 1 = 0$. This has $x = 1$ as a solution, but it is not a solution to the original equation. Factoring off $(x-1)$ yields $x^2 - 3x + 1 = 0$. The roots of this equation are $x = (3 \pm \sqrt{5})/2$. Since the original equation says that $x - 1$ is a square root, hence non-negative, we cannot have $x = (3 - \sqrt{5})/2$. The only remaining possibility is $x = (3 + \sqrt{5})/2 = ((1 + \sqrt{5})/2)^2$. It is easy to chack that this is a solution. The answer is **(b)**.

23. Since x degrees is the same as $\pi x/180$ radians, in terms of radians, we have $\cos(\pi x/180) = y$ and $\cos(x) = -y$. This means that $\pi - x = \pi x/180$ (plus multiples of 2π, which can be ignored here). Solving yields $x = 180\pi/(180 + \pi)$. The answer is **(e)**.

24. There are three possible first edges for the path and then 2 choices for the second edge. Label the vertices $\begin{matrix} A & B \\ C & D \end{matrix}$ on the top of the cube and $\begin{matrix} E & F \\ G & H \end{matrix}$ on the lower level (with A above E, etc.). Suppose that the bug goes from A to B to C (A to B to F is similar). If the bug next goes to D, it is forced to go to H, G, F, E, A. If the bug goes to G after C, then it is forced to go to F, E, H, D, A. Therefore, after the first two edges, there are exactly two ways to finish the trip. Since there are 6 choices for the initial two steps, there are 12 choices overall. The answer is **(c)**.

25. Since $\cos(a) + \cos(b) = 2\cos((a+b)/2)\cos((b-a)/2)$, we have

$$\cos x + \cos 4x + \cos 2x + \cos 3x = \cos(5x/2)\cos(3x/2) + \cos(5x/2)\cos(x/2)$$

$$= \cos(5x/2)\left(\cos(3x/2) + \cos(x/2)\right).$$

We have $\cos(5x/2) = 0$ for $x = \pi/5, 3\pi/5, \pi, 7\pi/5, 9\pi/5$. We have $\cos(3x/2) + \cos(x/2) = 0$ when $3x/2 = (2n+1)\pi - x/2$, which happens for $x = \pi/2, 3\pi/2$. There are therefore 7 values of x that make the expression 0. The answer is **(d)**.