THE 31$^{\text{st}}$ ANNUAL (2009) UNIVERSITY OF MARYLAND

HIGH SCHOOL MATHEMATICS COMPETITION

PART I MULTIPLE CHOICE

For each of the following 25 questions, carefully blacken the appropriate box on the answer sheet with a #2 pencil. Do not fold, bend or write stray marks on either side of the answer sheet. Each correct answer is worth 4 points. **Two points are deducted for each incorrect answer.** *Zero points are given if no box, or more than one box, is marked. Note that wild guessing is apt to lower your score. When the exam is over, give your answer sheet to the proctor. You may keep your copy of the questions.*

NO CALCULATORS

75 MINUTES

1. In this exam, suppose someone answers all 25 questions and receives a score of -8. How many questions does this person get correct? (*Hint:* Read the instructions for the exam.)
   a. 3 b. 5 c. 7 d. 9 e. 11

2. A dog named Toby goes to a supermarket. Toby has 2 dollars and is restricted to only two options: bones at 75 cents per pound and dry Puppy Chow at 25 cents per pound. Toby definitely prefers bones but must bring home at least 5 pounds of food. How many pounds of dry Puppy Chow will Toby buy? (The number of pounds is not required to be a whole number.)
   a. 3.5 b. 4.0 c. 4.5 d. 5.0 e. 5.5

3. Which of the following numbers is largest?
   a. $4^{(2^3)}$ b. $4^{(3^2)}$ c. $3^{(4^2)}$ d. $2^{(4^3)}$ e. $2^{(3^4)}$

4. One leg of a right triangle has length 15. The area of the triangle is 60. What is the length of the hypotenuse?
   a. 4 b. 8 c. 17 d. $\sqrt{241}$ e. $15\sqrt{2}$

5. At an elementary school, some children ride tricycles (which have three wheels) and the remaining children ride bicycles (which have two wheels). There is a total of 270 wheels at the school, and the number of children who ride bicycles is three times the number who ride tricycles. How many children are at the school?
   a. 80 b. 100 c. 108 d. 116 e. 120

6. Presidential elections occur during years that are multiples of 4. How many presidential elections (from 1788 through 2008) occurred during years that are perfect squares?
   a. 1 b. 2 c. 3 d. 4 e. 5

7. Let $n = 1 \times 3 \times 5 \times 7 \times \cdots \times 2009$ (so n is the product of the odd integers from 1 to 2009). What is the last digit of n?
   a. 1 b. 3 c. 5 d. 7 e. 9

8. A dog is searching for its owner, who is walking on the straight line that passes through the points $(1, 2)$ and $(3, 8)$. Poodle (the dog's version of Google) tells the dog to walk along the parabola $y = x^2 + x$. What is the x-coordinate of the point where their paths meet?
   a. -1 b. 0 c. 1 d. 3 e. their paths do not intersect

9. Napier's father asks him to go to the woodpile and carry in four logs. Napier returns with $\log_{10}(2)$, $\log_{10}(4)$, $\log_{10}(8)$, and $\log_{10}(16)$, and he piles them up as

$$\frac{\log_{10}(8) \cdot \log_{10}(16)}{\log_{10}(2) \cdot \log_{10}(4)}.$$

His father, much exasperated, asks, "What is the value of that?"
What is the value?
a. 1 b. $\log_{10}(2)$ c. $(\log_{10}(2))^4$ d. $6\,(\log_{10}(2))^2$ e. 6

10. The largest circle that fits inside an equilateral triangle T has area π. Each side of T has length
a. $\sqrt{3}$ b. $2\sqrt{3}$ c. $\pi\sqrt{3}$ d. $3\sqrt{3}$ e. $3\sqrt{3}/2$

11. In triangle ABC, let D be the intersection point of the bisector of angle ABC and the bisector of angle BCA. If angle CAB is $70°$, what is angle CDB?
a. $35°$ b. $55°$ c. $105°$ d. $125°$ e. $140°$

12. How many real numbers x with $0 < x \leq 10$ are solutions to $\log_{10}(x) = \sin(x)$, where x in $\sin(x)$ is in radians?
a. 0 b. 1 c. 2 d. 3 e. 4

13. Five people need to travel in a 5-passenger car. There are a driver's seat and a passenger seat in the front and three passenger seats in the back: a left seat, a middle seat, and a right seat. Two of the people are children and can sit only in the back. One of the three adults is busy reading a math book and refuses to drive. In how many ways can they get seated?
a. 6 b. 12 c. 18 d. 24 e. 30

14. Let p be the smallest prime number with 2009 digits. What is the remainder when p^2 is divided by 12?
a. 1 b. 3 c. 5 d. 7 e. 9

15. In a chess tournament each player plays one game with each of the other players. The winner of a game gets one point, the loser zero, and in case of a draw each player gets half a point. There were two juniors and a number of seniors playing in a high school chess championship. The total score of the juniors was 8 points, and each senior got the same score. There were at most 12 seniors. How many seniors played in the tournament?
a. 2 b. 6 c. 7 d. 10 e. 12

16. Let S be the set of all integers n with $1 \leq n \leq 2009$ such that 1 does not occur in the decimal expansion of n. How many integers are in S?
a. 1458 b. 1064 c. 737 d. 223 e. 222

17. Hilbert's daughter asks her father for the keys to the car 100 times. On the kth request, where $1 \leq k \leq 100$, Hilbert forms the product

$$P = (1+x)^k(1-x)^{100-k}$$

and expands the product to get a polynomial $P = 1 + ax + bx^2 + cx^3 + ...$ for some integers $a, b, c, \ldots$. If $b \leq 0$, Hilbert's answer is negative (that is, she doesn't get the keys) and if $b > 0$, Hilbert's answer is positive. How many times does she get the keys to the car?
a. 10 b. 49 c. 50 d. 89 e. 98

18. Let S_1 be a square with side s and C_1 be the circle inscribed in it. Let C_2 be a circle with radius r and S_2 be the square inscribed in it. We are told that the area of $S_1 - C_1$ is the same as the area of $C_2 - S_2$. Which of the following numbers is closest to s/r?
a. 1 b. 2 c. 3 d. 4 e. 5

19. Archimedes, Bernoulli, Cauchy, Dirichlet, and Euclid each choose a large prime number (they each choose a different prime).
Archimedes says, "My prime is not the largest and not the smallest."
Bernoulli says, "My prime is not the largest and not the smallest."
Cauchy says, "My prime is the largest."
Dirichlet says, "My prime is the smallest."
Euclid says, "My prime is the not the smallest."
Exactly one of the five mathematicians is lying. The others are telling the truth. Who has the largest prime?
a. Archimedes b. Bernoulli c. Cauchy d. Dirichlet e. Euclid

20. Suppose $f(x)$ is a function that satisfies $f(x) + 5f(1/x) = 3 + x$ for all non-zero real numbers x. What is $f(4)$?
a. 5/4 b. 1 c. 2 d. 27/83 e. 37/96

21. A ladder of length $\sqrt{15}$ meters is resting against a vertical wall. This creates a right triangle T with the ladder as the hypotenuse and the wall and the ground as the legs. There is a point on the ladder that is exactly 1 meter from the ground and exactly 1 meter from the wall. What is the sum of the lengths of the two legs of triangle T?
a. 3 b. $1 + \sqrt{5}$ c. $5 - \sqrt{2}$ d. 5 e. $(5 + \sqrt{5})/2$

22. Suppose y is a real number such that $2 < y < 3$ and $y^3 - 4y - 5 = 0$. What is the closest integer to y^2?
a. 4 b. 5 c. 6 d. 7 e. 8

23. A natural number is *amenable* if the sum of its digits is equal to the product of its digits. How many amenable 5-digit numbers are there? (*Note:* 00000 is not a 5-digit number.)
a. 10 b. 20 c. 30 d. 40 e. 50

24. How many real numbers x are solutions of

$$x = \left(x - \frac{1}{x}\right)^{1/2} + \left(1 - \frac{1}{x}\right)^{1/2} \quad ?$$

a. 0 b. 1 c. 2 d. 3 e. 4

25. A permutation $a_1, a_2, \ldots, a_{10}$ of the integers $1, 2, \ldots, 10$ is called *good* if $a_i < a_{i+2}$ for all i with $1 \leq i \leq 8$ and $a_j < a_{j+3}$ for all j with $1 \leq j \leq 7$. How many good permutations are there? (A permutation is a rearrangement of these integers. The trivial permutation $1, 2, \ldots, 10$ is counted as a permutation and, of course, it is good.)
a. 86 b. 87 c. 88 d. 89 e. 90