THE 31st ANNUAL (2009) UNIVERSITY OF MARYLAND
HIGH SCHOOL MATHEMATICS COMPETITION
PART I SOLUTIONS

1. If c is the number of correct answers, then $25 - c$ is the number of incorrect answers. The score is $-8 = 4c - 2(25 - c) = 6c - 50$. This yields $c = 7$. The answer is **c**.

2. Let b be the weight of bones and d the weight of dry dog food. Then $75b+25d = 200$ and $b+d \geq 5$. The first equation yields $b = (8 - d)/3$, so $(8 - d)/3 + d \geq 5$. This says that $d \geq 3.5$. Since Toby prefers bones, he brings home the minimum amount of dry dog food. The correct answer is **a**. (*Note:* Several students mistakenly thought that bones are not food and therefore assumed that Toby had to bring home at least 5 pounds of Puppy Chow. The question of how many pounds of Puppy Chow are bought then of course has 5 as the answer. We therefore decided also to accept **d** as an answer.)

3. The numbers can be rewritten as $2^{16}, 2^{18}, 3^{16}, 2^{64}, 2^{81}$. The last of these is clearly the largest ($2^{81} > 4^{16} > 3^{16}$). The answer is **e**.

4. Let x be the other leg. The area is $60 = \frac{1}{2}(x)(15)$, so $x = 8$. The hypotenuse is $\sqrt{15^2 + 8^2} = 17$. The answer is **c**.

5. Let t be the number of tricycles and b be the number of bicycles. Then $3t + 2b = 270$ and $b = 3t$. Therefore, $3t + 2(3t) = 270$, so $t = 30$. This means that $b = 3t = 90$, so there are 120 students at the school. The answer is **e**.

6. It must be the square of an even number. We have $42^2 = 1764$, $44^2 = 1936$, and $46^2 = 2116$. Only 1936 is in the required set of years. The answer is **a**.

7. n is odd and a multiple of 5. Therefore, it ends in 5. The answer is **c**.

8. The line is $y = 3x - 1$. This intersects the parabola when $3x - 1 = x^2 - x$, which rearranges to $0 = x^2 - 2x + 1 = (x - 1)^2$. Therefore, $x = 1$. The answer is **c**.

9. Write $8 = 2^3, 16 = 2^4, 4 = 2^2$. The expression becomes

$$\frac{3\log_{10}(2) \times 4\log_{10}(2)}{\log_{10}(2) \times 2\log_{10}(2)} = 6.$$

   The answer is **e**.

10. Let ABC be the triangle, let O be the center of the circle, and let M be the midpoint of BC. Then triangle OMB is a 30-60-90 triangle. Since the circle has area π, its radius is 1, so OM has length 1. Therefore, MB has length $\sqrt{3}$, so BC has length $2\sqrt{3}$. The answer is **b**.

11. The sum of the angles in triangle ABC is 180, so $ABC + ACB = 180 - 70 = 110$. Since BD and CD bisect the angles, $DBC + DCB = \frac{1}{2}(ABC + ACB) = 55$. Since $DBC + DCB + BDC = 180$, we have $BDC = 180 - 55 = 125$. The answer is **d**.

12. Draw the graphs of $y = \log_{10}(x)$ and $y = \sin(x)$. The graphs cross once between 0 and π, and twice between 2π and 3π. The answer is **d**.

13. There are two choices for the driver. After the driver is seated, there are two choices remaining for the front-seat passenger. There are $3 \times 2 \times 1 = 6$ ways to seat the remaining passengers in the back. This gives $2 \times 2 \times 6 = 24$ total ways to seat the people. The answer is **d**.

14. When p is divided by 12, the remainder is 1, 5, 7, or 11 (the other possibilities give multiples of 2 or 3). Therefore, $p = 12j + k$, where $k = 1, 5, 7,$ or 11. Squaring yields $p^2 = 144j^2 + 24jk + k^2$, which is a multiple of 12 plus k^2. The value of k^2 is 1, 25, 49, or 121, each of which is one more than a multiple of 12. Therefore, p^2 is 1 more than a multiple of 12, so the remainder is 1. The answer is **a**.

15. Let s be the number of seniors, so the total number of players is $s + 2$. Each player plays $s + 1$ games, so the total number of games is $\frac{1}{2}(s + 1)(s + 2)$ (divide by 2 since $(s + 1)(s + 2)$ counts each game twice, once for each player). Each game yields a total of 1 point awarded (either $0 + 1$ or $\frac{1}{2} + \frac{1}{2}$). Therefore, the total number of points awarded is $\frac{1}{2}(s+1)(s+2)$. Let x be the common score of the seniors. Then $8 + x \times s = \frac{1}{2}(s + 1)(s + 2)$. This yields $s^2 + 3s - 2xs = 14$, which says that 14 is a multiple of s. So $s = 1, 2, 7, 14$. Since $s = 1, 2$ yields negative x and $s = 14$ is too large, we must $s = 7$ (and $x = 4$). The answer is **c**.

16. Since any digit except 1 can be used, there are $9^3 = 729$ integers from 0 to 999 that use no ones, hence 728 from 1 to 999. All of the integers from 1000 to 1999 use ones. From 2000 to 2009 there are 9 that do not use ones. The total is 737. The answer is **c**.

17. We can ignore powers of x higher than x^2. We have

$$P = (1 + kx + \frac{1}{2}k(k-1)x^2 + \cdots)(1 - (100 - k)x + \frac{1}{2}(100 - k)(99 - k)x^2 + \cdots).$$

The coefficient of x^2 is $\frac{1}{2}k(k-1) + \frac{1}{2}(100-k)(99-k) - k(100-k) = 2(k-45)(k-55)$. This is positive when $k \leq 44$ or $k \geq 56$, so there are 89 values of k that give a positive coefficient. The answer is **d**.

18. The area of $S_1 - C_1$ is $s^2 - (\pi/4)s^2$. The area of $C_2 - S_2$ is $\pi r^2 - 2s^2$. Therefore,

$$s^2/r^2 = 4\frac{\pi - 2}{4 - \pi} = \frac{8}{4 - \pi} - 4.$$

Since $3 < \pi < 3.2$, we have $8 < 8/(4 - \pi) < 10$. Therefore, $4 < s^2/r^2 < 6$, so $2 < s/r < 2.5$. The closest integer to s/r is 2, so the answer is **b**.

19. If C and D are telling the truth, then no one is lying. Therefore, either C or D is lying. If D is lying then someone else has the smallest, so one of A, B, C, E is also lying, which is not allowed. Therefore, C is lying and A, B, D, E are telling the truth. This allows only the possibility that E has the largest prime. The answer is **e**.

20. Let $a = f(4)$ and $b = f(1/4)$. Setting $x = 4$ yields $a + 5b = 7$. Setting $x = 1/4$ yields $b + 5a = 13/4$. Solving these two equations yields $a = 37/96$. The answer is **e**.

21. Let x be the vertical leg and y be the horizontal leg of the right triangle. Similar triangles yield the relation $(x - 1)/1 = 1/(y - 1)$, so $(x - 1)(y - 1) = 1$, hence $xy = x + y$. The Pythagorean theorem yields $x^2 + y^2 = 15$. Therefore,

$$(x + y)^2 = x^2 + y^2 + 2xy = 15 + 2(x + y).$$

This yields $(x + y - 5)(x + y + 3) = 0$, so $x + y = 5$. The answer is **d**.

22. Dividing by y yields $y^2 = 4 + (5/y)$. Since $2 < y < 3$, we have $5/3 < 5/y < 5/2$, so $4 + (5/3) < y^2 < 4 + (5/2)$. This means the closest integer to y^2 is 6. The answer is **c**.

23. The amenable numbers are the 10 distinct permutations of 22211, the 10 distinct permutations of 33111, and the 20 distinct permutations of 52111. This gives a total of 40 amenable numbers. The answer is **d**.

24. Let $a = \sqrt{x - 1/x}$ and $b = \sqrt{1 - 1/x}$. Then $a + b = x$ and $(a + b)(a - b) = a^2 - b^2 = x - 1$. Therefore, $a - b = (x - 1)/x$. Adding yields $2a = x + (x - 1)/x = 1 + a^2$, which says that $(a - 1)^2 = 0$. Therefore, $a = 1$, so $x - 1/x = 1$. This yields $x = (1 \pm \sqrt{5})/2$. Only $(1 + \sqrt{5})/2$ satisfies the original equation. The answer is **b**.

25. Let s_k be the number of good permutations of the integers from 1 to k. Clearly $s_1 = 1$ and $s_2 = 2$. It is easy to see that k must occur in one of the last 2 positions. If k occurs in the last position, then a good permutation is simply a good permutation of 1 to $k - 1$ with k appended. If k occurs in the next to last place, then $k - 1$ must occur in the last place. The first $k - 2$ numbers are therefore a good permutation of 1 to $k - 2$. It follows that $s_k = s_{k-1} + s_{k-2}$. Since $s_1 = 1$ and $s_2 = 2$, we see that the numbers s_k are Fibonacci numbers and $s_{10} = 89$. The answer is **d**.