THE 32nd ANNUAL (2010) UNIVERSITY OF MARYLAND

HIGH SCHOOL MATHEMATICS COMPETITION

PART I SOLUTIONS

1. We calculate $G = 2 \cdot 3 \cdot 5 = 30$, $H = 7 + 11 + 13 + 17 + 19 = 67$, and $E = 30 \cdot 67 = 2010$. The answer is **e**.

2. The area of the pizzas given are A) $5 \times \pi \times 4^2 = 80\pi$, B) $2 \times \pi \times 7^2 = 98\pi$, and C) $\pi \times 9^2 = 81\pi$. Since $80 < 81 < 98$, the answer is **b**.

3. The line intersects the axes at the points $(0, 1/4)$ and $(1/3, 0)$. The right triangle with vertices at these two points and $(0, 0)$ has sides of lengths $1/3$, $1/4$, and $\sqrt{1/9 + 1/16} = 5/12$. Since $1/3 + 1/4 + 5/12 = 1$, the answer is **e**.

4. The sum $11 + 15 + 14 = 40$ counts every candy eaten twice. Hence the total number of candies eaten is $40/2 = 20$. The answer is **b**.

5. The assumption implies that the equation $x^2 + 2ax = 2x - 4$ has no solutions in real numbers x. Rewrite this as $x^2+2(a-1)x+4 = 0$, a quadratic equation with discriminant $D = 4(a-1)^2-16 = 4(a - 3)(a + 1)$. The quadratic equation has no solutions if and only if $D < 0$, which happens if and only if $-1 < a < 3$. The answer is **a**.

6. We have eight 1×1 rectangles, ten 1×2 rectangles, four 1×3 rectangles, two 1×4 rectangles, three 2×2 rectangles, two 2×3 rectangles, and one 2×4 rectangle, for a total of 30 rectangles. The answer is **d**.

7. The distances travelled by the two bulls are $v_1 t$ feet and $v_2 t$ feet, respectively, and the sum of these two must equal d. Therefore $(v_1 + v_2)t = d$, and so $t = d/(v_1 + v_2)$. The answer is **a**.

8. The equation becomes $(x^2 - 1)^2 = 0$, which is equivalent to $x^2 - 1 = 0$. The latter has the two solutions $x = 1$ and $x = -1$. The answer is **c**.

9. If $S = 1 + 3 + \cdots + 99$, then

$$2S = (1 + 3 + \cdots + 99) + (99 + 97 + \cdots + 1) = (1 + 99) + (3 + 97) + \cdots + (99 + 1) = 50 \cdot 100,$$

so $S = 50^2$. Also, we have $2+4+\cdots+100 = S+50 = 50 \cdot 51$, and therefore the required fraction is $S/(S + 50) = 50/51$. The answer is **b**.

10. Exactly one of the logicians is telling the truth, because their statements are pairwise incompatible, and all five cannot be lying as then Cinco would be lying and telling the truth at the same time. Since exactly four of them are lying, only Quatro is telling the truth. The answer is **d**.

11. We must compare the numbers $\log(20^{1/2})$, $\log(30^{1/3})$, $\log(40^{1/4})$, $\log(50^{1/5})$, and $\log(60^{1/6})$. Since log is an increasing function, this is equivalent to comparing the numbers $20^{1/2}$, $30^{1/3}$,$40^{1/4}$, $50^{1/5}$, and $60^{1/6}$. Now $20 > 4^2 = 16$, so $20^{1/2} > 4$. Also, $30 < 4^3 = 64$, so $30^{1/3} < 4$, and similarly $40^{1/4}$, $50^{1/5}$, and $60^{1/6}$ are each less than 4. The answer is **a**.

12. We have $\cos\theta = \sin\theta/\cos\theta$, or $\cos^2\theta = \sin\theta$. Equivalently, $1 - \sin^2\theta = \sin\theta$. Setting $\sin\theta = x$, we deduce that x is a root of the quadratic equation $x^2 + x - 1 = 0$. This has two solutions, and the only positive one is $x = (\sqrt{5} - 1)/2$. The answer is **e**.

13. Assume that the cabin is at point A, the point one mile away from A is B, and the tip of the mountain top is at point C. Suppose also that C projects onto a point M on the line AB such that $|CM| = h$. We obtain the equations $|AM| = h\cot\alpha$ and $|BM| = h\cot\beta$, so $h(\cot\alpha - \cot\beta) = |AM| - |BM| = |AB| = 1$. Therefore we have $h = 1/(\cot\alpha - \cot\beta) = \tan\alpha\tan\beta/(\tan\beta - \tan\alpha)$. The answer is **d**.

14. We may assume that $x > y > 0$ (otherwise replace one or both of x and y by their negatives). Now $(x-y)(x+y) = 100$, and since $(x-y)+(x+y) = 2x$ is even, the two factors $x-y$ and $x+y$ must both be even (they are not both odd, since their product 100 is even). The only way to factor 100 into two even factors, one less than the other, is $100 = 2 \cdot 50$. We deduce that $x - y = 2$ and $x + y = 50$, from which we obtain $x = 26$ and $y = 24$. Therefore, $x^2 + y^2 = 26^2 + 24^2 = 1252$. The answer is **a**.

15. Three numbers a, b, and c with $a < b < c$ can be the lengths of the sides of a triangle if and only if the inequality $a+b > c$ holds. There are seven triples (a, b, c) among the six sticks which satisfy this, namely, $(2, 3, 4)$, $(2, 4, 5)$, $(2, 5, 6)$, $(3, 4, 5)$, $(3, 4, 6)$, $(3, 5, 6)$, and $(4, 5, 6)$. The answer is **b**.

16. For any real number x, we have

$$f(x) + 3 \leq f(x+3) = f((x+2)+1) \leq f(x+2) + 1 \leq f(x+1) + 2 \leq f(x) + 3.$$

Since the two ends of the above sequence of inequalities are identical, all of the inequalities must in fact be equalities. In particular, we have $f(x+1) + 2 = f(x) + 3$, or $f(x+1) = f(x) + 1$, for all $x \in \mathbb{R}$. By induction on n, it follows that $f(x+n) = f(x) + n$ for all positive integers n. For $x = 1$ and $n = 2009$ this gives $f(2010) = f(1) + 2009 = 2009$. The answer is **c**.

17. Since $\dfrac{1}{y + \frac{1}{z}} < 1$ and x is a positive integer, we must have $x = \lfloor \frac{8}{3} \rfloor = 2$. It follows that $y + \dfrac{1}{z} = \dfrac{3}{2}$, and by similar reasoning we see that $y = \lfloor \frac{3}{2} \rfloor = 1$, and $z = 2$. The answer is **b**.

18. The sequence begins with 9 single digit numbers, followed by 90 two digit numbers. Hence we reach the second '9' in 99 after $9 + 90 \cdot 2 = 189$ digits. There are $1821 = 2010 - 189$ more digits to get to 2010, and since $1821 = 3 \cdot 607$ this means 607 more three digit numbers. As $99 + 607 = 706$, the 2010-th digit in the sequence is the last digit of 706, namely 6. The answer is **c**.

19. The perpendicular cross section of the tube is a circle K of radius 2. Let O be the center of K, OD be a radius of K, and AB be the chord of K that is perpendicular to OD and intersects OD at its midpoint M. Let x be the area of the region – a circular segment – bounded by the chord AB and arc ADC of K. Since the volume of a cylindrical region of space is the area of its base times its height, the required fraction is the equal to x divided by the area 4π of K itself. The right triangle OMA has $|OA| = 2$ and $|OM| = 1$, hence angle OAM equals $30°$ and angle AOB equals $2 \cdot (90° - 30°) = 120°$. If DO intersects K at a second point C, so that CD is a diameter of K, then ABC is an equilateral triangle, and the area 4π of K is therefore equal to $3x + \text{Area}(ABC) = 3x + 3\sqrt{3}$. We conclude that $x/(4\pi) = 1/3 - \sqrt{3}/(4\pi) = (4\pi - 3\sqrt{3})/(12\pi)$. The answer is **e**.

20. We have $b = a^{3/2}$ and hence $a^3 = b^2$, while $d = c^{5/4}$ and hence $c^5 = d^4$. Since a, b, c, and d are positive integers, we must have $a = x^2$ and $c = y^4$ for some positive integers x and y. Now $9 = a - c = x^2 - y^4 = (x - y^2)(x + y^2)$, so $x - y^2 = 1$ and $x + y^2 = 9$. Solving this system gives $x = 5$ and $y = 2$. Finally, $b - d = x^3 - y^5 = 125 - 32 = 93$. The answer is **d**.

21. The four people among the eight who get dealt a card that matches the first letter of their name can be chosen in $\binom{8}{4} = 70$ ways. We have to multiply this by the number of ways to permute the remaining four cards so that there are no matches. This is the same as the number of permutations (a_1, a_2, a_3, a_4) of $(1, 2, 3, 4)$ so that $a_i \neq i$ for $1 \leq i \leq 4$. There are exactly 9 of these, namely $(2, 1, 4, 3)$, $(3, 1, 4, 2)$, $(4, 1, 2, 3)$, $(2, 4, 1, 3)$, $(3, 4, 1, 2)$, $(4, 3, 1, 2)$, $(2, 3, 4, 1)$, $(3, 4, 2, 1)$, and $(4, 3, 2, 1)$. The answer is $70 \cdot 9 = 630$, so **d**.

22. We have

$$10^{2010} + 1 = 100^{1005} + 1^{1005} = (100 + 1)(100^{1004} - 100^{1003} + 100^{1002} - \cdots - 100 + 1).$$

We deduce that the smallest integer greater that $10^{2010}/101$ is $100^{1004} - 100^{1003} + \cdots - 100 + 1 = 100k + 1$, for some positive integer k. Since the units digit of this number is 1, the answer is **b**.

23. Since a, b, and c are the roots of the polynomial $x^3 - 3x + 1$, we have a factorization

$$x^3 - 3x + 1 = (x - a)(x - b)(x - c) = x^3 - (a + b + c)x^2 + (ab + bc + ca)x - (abc).$$

Equating the coefficients of like powers of x in the above gives $a + b + c = 0$, $ab + bc + ca = -3$, and $abc = -1$. It follows that

$$\begin{aligned}\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} &= \frac{a^2b^2 + b^2c^2 + c^2a^2}{(abc)^2} = a^2b^2 + b^2c^2 + c^2a^2 \\ &= (ab + bc + ca)^2 - 2abc(a + b + c) = (-3)^2 = 9.\end{aligned}$$

The answer is **e**.

24. Suppose that the vertices of ABC and the six division points on its three sides are A, D, E, B, F, G, C, H, and I in counterclockwise order, so that $|DE| = |FG| = |IH| = x$. Since triangle DEM is similar to ABC, we have $|DE|/|AB| = |EM|/|BC| = |BF|/|BC|$. We similarly see that $|IH|/|AC| = |MH|/|BC| = |GC|/|BC|$. We deduce that

$$\frac{13x}{12} = \frac{x}{2} + \frac{x}{3} + \frac{x}{4} = \frac{|DE|}{|AB|} + \frac{|FG|}{|BC|} + \frac{|IH|}{|AC|} = \frac{|BF|}{|BC|} + \frac{|FG|}{|BC|} + \frac{|GC|}{|BC|} = \frac{|BC|}{|BC|} = 1,$$

and hence $x = 12/13$. The answer is **c**.

25. Since $4^{37} + 4^{1000} + 4^n = 2^{74}(1 + 2 \cdot 2^{1925} + 2^{2n-74})$, and $2^{74} = (2^{37})^2$ is a perfect square, the problem is equivalent to finding the largest integer n such that $S := 1 + 2 \cdot 2^{1925} + 2^{2n-74}$ is a perfect square. Observe that S is equal to the square $(1 + 2^{1925})^2$ when $2n - 74 = 2 \cdot 1925$, which happens when $n = 1925 + 37 = 1962$. If $n > 1962$, then

$$(2^{n-37})^2 < 1 + 2 \cdot 2^{1925} + (2^{n-37})^2 = S < (1 + 2^{n-37})^2,$$

therefore S lies between the squares of two consecutive natural numbers, and hence cannot be a perfect square. The answer is **c**.