THE 33rd ANNUAL (2011) UNIVERSITY OF MARYLAND

HIGH SCHOOL MATHEMATICS COMPETITION

PART I SOLUTIONS

1. The cashier gives us four 13 cent coins and six 8 cent coins, for a total of $(4 \cdot 13) + (6 \cdot 8) = 100$. Since we received $4 + 6 = 10$ coins, the answer is **c**.

2. If the lines intersect at the point with coordinates (x_0, y_0), then $13x_0 - 1 = 10x_0 + b$, so $3x_0 = b+1$ and $x_0 = (b+1)/3$. It follows that
$$y_0 = 10x_0 + b = \frac{10(b+1)}{3} + b = \frac{13b+10}{3}.$$
Therefore $y_0 = 1000$ if $13b + 10 = 3000$, or $b = 2990/13 = 230$. The answer is **a**.

3. 500 students got Question 1 wrong, and 800 students got Question 2 wrong. The number of students that got either Question 1 or Question 2 wrong is $500 + 800 - 100 = 1200$, since the sum $500 + 800$ counts the 100 students that got both questions wrong twice. Therefore, the number of students answering both questions correctly is $2000 - 1200 = 800$. The answer is **d**.

4. If $x - y = 2$ and $xy = 10$, then $x = y + 2$ and hence $(y + 2)y = 10$, or $y^2 + 2y - 10 = 0$. This quadratic equation has the two solutions $\sqrt{11} - 1$ and $-\sqrt{11} - 1$, but only the former is positive. Hence $y = \sqrt{11} - 1$ and $x = y + 2 = \sqrt{11} + 1$. The answer is **d**.

5. Suppose that the baseball team won n games in the season, and so played $9n$ games in total. We are told that $n - 1 = (9n - 1)/10$, or $10n - 10 = 9n - 1$, which implies that $n = 9$, for a total of 81 games played in the season. The answer is **e**.

6. We have $AM = BM = CM$, so both triangles AMB and AMC are isosceles. It follows that $\angle BAM = \angle ABM = 40^\circ$ and $\angle AMC = \angle BAM + \angle ABC = 80^\circ$. Since $\angle AMC + 2(\angle ACB) = 180^\circ$, we deduce that $\angle ACB = 50^\circ$. The answer is **d**.

7. Using the property given for f, we see that $f(2)+f(5) = f(10) = 6$ and $2f(2)+f(5) = f(20) = 10$. The first equation implies that $2f(2) + 2f(5) = 12$, and subtracting the second equation from this gives $f(5) = 12 - 10 = 2$. We deduce that $f(25) = f(5) + f(5) = 4$, so the answer is **a**.

8. We have $\log(\sqrt[3]{10}) = \log(10^{\frac{1}{3}}) = \frac{1}{3}\log(10) = \frac{1}{3} = 0.3333.....$ The answer is **b**.

9. We have $\sin(45^\circ) + \cos(45^\circ) = \sqrt{2}$, $\sin(60^\circ) + \cos(60^\circ) = (\sqrt{3} + 1)/2$, $\sin(90^\circ) + \cos(90^\circ) = 1$, $\sin(120^\circ) + \cos(120^\circ) = (\sqrt{3} - 1)/2$, and $\sin(135^\circ) + \cos(135^\circ) = 0$. The first two numbers are greater than 1, and the last two are less than 1. Observe that
$$\left(\frac{\sqrt{3}+1}{2}\right)^2 = \frac{4+2\sqrt{3}}{4} = 1 + \frac{\sqrt{3}}{2} < 1 + \frac{\sqrt{4}}{2} = 2 = (\sqrt{2})^2.$$
We deduce that $(\sqrt{3} + 1)/2 < \sqrt{2}$, and therefore the answer is **a**.

10. If the sides of the rectangle have lengths x and y, then the assumptions give $xy = a$ and $x^2+y^2 = d^2$. It follows that $(x + y)^2 = x^2 + y^2 + 2xy = d^2 + 2a$ and hence $x + y = \sqrt{d^2 + 2a}$. Since the perimeter equals $2(x + y)$, the answer is **e**.

11. Suppose that Cathy had a apples. Then Bob has $2a$ apples, and Anna has $4a$ apples, so in total the three have $7a$ apples. Now suppose that Anna has b oranges. Then Cathy has $3b$ oranges, and Bob has $9b$ oranges, so together the three have $13b$ oranges. We are told that $7a = 13b$. Since 7 and 13 are distinct primes, 7 must divide b, and hence $b = 7k$ for some positive integer k. Therefore $13b = 91k$, and the total number of apples and oranges the group picked is $182k$. Since we are told that this number is less than 250, we have $k = 1$ and the answer is **b**.

12. Suppose that the speed of the rower in a lake without current is v, the speed of the river current is v', and the distance between points A and B is d. The assumptions imply that $d = 2(v+v') = 4(v-v')$, and hence $2v+2v' = 4v-4v'$, or $v' = v/3$. We deduce that $d = 2(v+v/3) = 8v/3$ and $d/v = 8/3$. Since d/v equals the time required to go distance d rowing at a constant speed v, we get $2+2/3$ hours, or 2 hours and 40 minutes. The answer is **b**.

13. Let $f(n) = 1+2+\cdots+n = n(n+1)/2$. We seek n such that $f(n) \leq 2011 < f(n+1)$, or equivalently $n^2+n \leq 4022 < n^2+3n+2$. Since n is close to the square root of 2011, we find that $n = 62$. On the last day we therefore have $2011 - (31 \cdot 63) = 58$ kernels of corn left on the plate. The answer is **d**.

14. Suppose that A is lying. Then C is telling the truth, so B is a Hood. But if B is telling the truth, then A must be a Hood, which contradicts the fact that A is lying. Therefore A must be a Hood, C is a Wolf, and hence B is also a Wolf. The answer is **e**.

15. The quantity $|x+12| + |x-5|$ equals the sum of the distances from the point x to the points -12 and 5 on the real number line. For any x such that $-12 \leq x \leq 5$, this sum of distances is $5-(-12) = 17$. Therefore $r = 17$ gives infinitely many solutions and the answer is **d**.

16. Suppose that each side of ABC has length a. Since the area 1 of ABC equals $a^2\sqrt{3}/4$, we obtain that $a^2 = 4/\sqrt{3}$. The sum of the areas of the three erected squares on the sides of ABC is therefore $3a^2 = 12/\sqrt{3} = 4\sqrt{3}$. The hexagon constructed in the problem consists of these 3 squares together with triangle ABC plus three other isosceles triangles, each having two sides of length a meeting at an angle of 120°. These three triangles have area equal to the area of ABC; to see this, one can use the formula $a^2 \sin(120°)/2$ for the area and note that $\sin 120° = \sin 60°$. Summing all the areas up gives $4+4\sqrt{3}$, and the answer is **e**.

17. The first few applications of the machine give as output $2n+1$, $4n+3$, $8n+7$, $16n+15$, etc. It appears as if applying the machine k times results in $2^k n + (2^k - 1)$, and indeed this is the case (for an easy proof, one can use induction on k). The assumption of the problem therefore gives $2^{12}n + (2^{12}-1) = 827391$, from which we get $n = (827391-4095)/4096 = 201$. The answer is **c**.

18. If a valid game of tic-tac-toe fills up the whole board, then X must have played the last move. We deduce that O did not win, as the game would have stopped earlier. This gives the only impossible configurations, i.e., those for which there are three O's in the same row, column, or diagonal. There are 3 rows, 3 columns, and 2 diagonals, so 8 possibilities for the line of 3 O's. For each of these 8, there are 6 possible positions for the fourth O. This gives a total of $8 \times 6 = 48$ illegal end positions, and so $78 = 126-48$ legal ones. The correct answer 78 was not among those given. Since this was our mistake, we awarded 4 points to every student for problem 18.

19. The number $416183 = 3 \cdot 138727 + 2$ leaves a remainder of 2 upon division by 3. If we divide p and q by 3, the possible remainders are 1 or 2 for each. But these two remainders must be different, as otherwise the product pq would leave remainder 1 upon division by 3. Indeed, suppose e.g.

that $p = 3a + 2$ and $q = 3b + 2$ for some integers a and b. Then $pq = (3a + 2)(3b + 2) = 3(3ab + 2a + 2b + 1) + 1$; the argument if the common remainder equals 1 is similar. It follows that $p + q$ leaves a remainder of $1 + 2 = 3$ when we divide it by 3, and so is divisible by 3. The only number among the answers that is a multiple of 3 is 1296. The answer is **c**.

20. If we divide both sides of the equation by 3^{2x} and set $y = \left(\frac{5}{3}\right)^x$, we obtain $y^2 - \frac{34}{15}y + 1 = 0$. If we set $\frac{17}{15} = 1 + c$, where $c = \frac{2}{15}$, then the roots of the equation in y are $y_{1,2} = 1 + c \pm \sqrt{c^2 + 2c}$. As $1 + c > \sqrt{c^2 + 2c}$, both y_1 and y_2 are positive. We conclude that $x_1 = \log_{\frac{5}{3}}(y_1)$ and $x_2 = \log_{\frac{5}{3}}(y_2)$ are the two solutions of the original equation. The answer is **c**.

21. If $F = 0$ then we must have $E = G = 0$ as well, which contradicts the assuption that all the integers are distinct. Therefore $F > 0$ and the equation $EF = F$ implies that $E = 1$. We also get $F = G + 1$ and since $G \notin \{0, 1\}$, we deduce that $F \geq 3$. If $F = 3$ and $G = 2$ then $D \geq 4$ and $C = 3D \geq 12$, a contradiction. Similarly we cannot have $F \geq 5$, so we get $F = 4$ and $G = 3$. As $C = 4D$ we must have $C = 8$, $D = 2$ and hence $A = 9$, $B = 5$. Finally $A + B + C + D + E + F + G = 1 + 2 + 3 + 4 + 5 + 8 + 9 = 32$ and the answer is **b**.

22. We claim first that the number of solutions to the equation

$$x_1 + x_2 + x_3 = 1001 \tag{1}$$

where the x_i are (unordered) positive integers is 499500. To see this, consider 1001 points marked linearly along a stick. There are 1000 'gaps' (i.e. open intervals) between any two successive points among these 1001. Each solution of (1) corresponds to dividing the stick into three pieces by cutting it at two of these 1000 gaps, so that x_i equals the number of points on the i-th piece, for $i = 1, 2, 3$. The number of possible ways to choose 2 gaps among the 1000 is the binomial coefficient $\binom{1000}{2} = 1000 \cdot 999/2 = 499500$, as claimed. Now 1001 is not a multiple of three, so we cannot have $x_1 = x_2 = x_3$ – however two of the x_i can be equal. The equation $2x + y = 1001$ has 500 solutions (x, y) in positive integers (y must be odd, so $y \in \{1, 3, 5, \ldots, 999\}$, and for each of these 500 choices for y we get $x = (1001 - y)/2$). Each such pair (x, y) corresponds to three solutions of (1), with one of the x_i equal to y and the remaining two equal to x. So we have $499500 - 1500 = 498000$ solutions of (1) in triples (x_1, x_2, x_3) of *distinct* positive integers x_i. The number of solutions (a, b, c) with $a < b < c$ is obtained from these by dividing by 6, as there are 6 possible ways to permute three distinct integers a, b, and c. The final answer is therefore $498000/6 = 83000$, so **b**.

23. Rewrite the given equation as $\sin^2 3x - \sin^2 x = \sin 3x \cos x - \sin x \cos 3x = \sin 2x$ and observe that

$$\sin^2 3x - \sin^2 x = (\sin 3x + \sin x)(\sin 3x - \sin x) = (2 \sin 2x \cos x)(2 \cos 2x \sin x) = \sin 4x \sin 2x.$$

Therefore the original equation becomes $(\sin 2x)(\sin(4x) - 1) = 0$, which gives $\sin 2x = 0$ or $\sin 4x = 1$. The first equation has the four solutions $\{0, \pi/2, \pi, 3\pi/2\}$ in $[0, 2\pi)$ while the second has the four solutions $\{\pi/8, 5\pi/8, 9\pi/8, 13\pi/8\}$ in the interval $[0, 2\pi)$. Adding these eight numbers gives $13\pi/2$. The answer is **d**.

24. Suppose that $m = 7\text{abcdefg}77 = n^3$, where $n = \text{wxyz}$ and a, b, c, d, e, f, g, w, x, y, z all denote digits. We have $1900^3 = 6859000000 < m < 2000^3$, hence $1900 < n < 2000$ and so $n = 19\text{yz}$. Clearly m and n are both odd and not multiples of 5, so $\text{z} \in \{1, 3, 7, 9\}$. Now 3^3 is the only

number in $\{1^3, 3^3, 7^3, 9^3\}$ that ends in 7, hence $\mathrm{z} = 3$ and $n = 19\mathrm{y}3$. Finally, the last two digits 77 of n^3 are the same as the last two digits of $(10\mathrm{y} + 3)^3$, so the difference

$$(10\mathrm{y} + 3)^3 - 77 = (1000\mathrm{y}^3 + 900\mathrm{y}^2 + 270\mathrm{y} + 27) - 77 = 100(10\mathrm{y}^3 + 9\mathrm{y}^2) + 10(27\mathrm{y} - 5)$$

must be a multiple of 100. This happens if and only if 10 divides $27\mathrm{y} - 5$, therefore $\mathrm{y} = 5$, $n = 1953$, and by computing 953^3 (or $(-47)^3$) we find that $\mathrm{g} = 1$. The answer is **a**.

25. Observe that if $z = x + y + xy$, then $z + 1 = (x + 1)(y + 1)$. Therefore instead of the numbers written on the blackboard, it is helpful to consider the numbers which are obtained adding 1 to each of them. In terms of the latter integers, each new number written on the board will be the product of two numbers which are already there. Initially we had the numbers $2 = 1 + 1$ and $3 = 2 + 1$ on the board, so all numbers that followed must be of the form $2^m 3^n$ for some nonnegative integers m and n. Among the given numbers, we have $1727 + 1 = 12^3 = 2^6\,3^3$, $2303 + 1 = 48^2 = 2^8\,3^2$, $2467 + 1 = 2^2 \cdot 617$, $3455 + 1 = 1728 \cdot 2 = 2^7\,3^3$, and $7775 + 1 = 6^5 = 2^5\,3^5$. Therefore the answer is **c**.