THE 34th ANNUAL (2012) UNIVERSITY OF MARYLAND

HIGH SCHOOL MATHEMATICS COMPETITION

PART I SOLUTIONS

1. After Halloween, the costumes cost $200 - 200/10 = 180$ dollars each. After Thanksgiving, the price is lowered to $180 - 180/5 = 144$ dollars. The answer is **b**.

2. The only single digit primes are 2, 3, 5, and 7. These four digits can be used to form the four 2-digit primes 23, 37, 53, and 73. The answer is **b**.

3. The inequality $|n - 2012| < 34$ is equivalent to $-34 < n - 2012 < 34$, or $1978 < n < 2046$. The answer is **c**.

4. We have
$$\frac{1}{1+\frac{1}{1+\frac{1}{1+1}}} = \frac{1}{1+\frac{1}{1+\frac{1}{2}}} = \frac{1}{1+\frac{1}{\frac{3}{2}}} = \frac{1}{1+\frac{2}{3}} = \frac{1}{\frac{5}{3}} = \frac{3}{5}.$$
The answer is **d**.

5. The integer $2012d2012$ is a multiple of 9 if and only if the sum of its digits is a multiple of 9. Therefore $d + 10$ must be a multiple of 9, and we deduce that $d = 8$. The answer is **e**.

6. The hypotenuse has length $4^{10}\sqrt{2} = 4^{10}4^{1/4} = 4^{10.25}$. The answer is **a**.

7. The parabola intersects the coordinate axes at the three points $(-\frac{1}{\sqrt{a}}, 0)$, $(0, -1)$, and $(\frac{1}{\sqrt{a}}, 0)$, which are the vertices of an isosceles triangle. The triangle will be equilateral if and only if the length $\frac{2}{\sqrt{a}}$ of its base is equal to the length $\sqrt{1+\frac{1}{a}}$ of its lateral sides. Squaring both quantities leads to the equation
$$\frac{4}{a} = 1 + \frac{1}{a} = \frac{a+1}{a}$$
which simplifies to $a^2 - 3a = 0$. Since $a > 0$, the unique answer is $a = 3$, or **e**.

8. We compute that there must have been $\frac{1}{2}(6 + 7 - 9) = 2$ completely clear days. Therefore, there were $9 + 2 = 11$ days in the period. The answer is **e**.

9. The last digit of 2012^{2013} is the same as the last digit of 2^{2013}. The last digit of 2^k for $k = 1, 2, \ldots$ is $2, 4, 8, 6, 2, 4, 8, 6, \ldots$, which is a periodic sequence with period 4. Since $2013 = 4 \cdot 503 + 1$, it follows that the last digit of 2^{2013} is 2. The answer is **a**.

10. Suppose that the seven small rectangles have parallel sides of lengths x and y with $x < y$. We are given the equality of areas $7xy = 336$, or equivalently $xy = 48$. Moreover, comparing x, y with the long side of $ABCD$ gives $4x = 3y$, or $x = 3y/4$. Substituting this value of x into the first equation and solving for y gives the unique solution $x = 6$, $y = 8$. We conclude that the perimeter of $ABCD$ is $6x + 5y = 36 + 40 = 76$. The answer is **a**.

11. We compute that
$$1000^{\log 2+\log 3} = 1000^{\log 2}1000^{\log 3} = (10^3)^{\log 2}(10^3)^{\log 3} = (10^{\log 2})^3(10^{\log 3})^3 = 2^3 3^3 = 8 \cdot 27 = 216.$$
The answer is **c**.

12. Let v_A, v_B, and v_C be the velocities of runners A, B, and C, respectively, and t_A, t_B, and t_C be the times they took to complete the 100 meter race. We are given that $v_At_A = v_Bt_B = 100$, while $v_At_A - v_Bt_A = 10$ and $v_Bt_B - v_Ct_B = 10$. It follows that $v_Bt_A = v_Ct_B = 90$, and multiplying these two equations gives $v_Ct_Av_Bt_B = 8100$, and hence $v_Ct_A = 81$. Finally, we are asked to calculate $v_At_A/2 - v_Ct_A/2 = 50 - v_Ct_A/2$, which equals 9.5 meters. The answer is **b**.

13. Since the angles are measured in radians and $1 \in (0, \frac{\pi}{2})$ while $2, 3 \in (\frac{\pi}{2}, \pi)$, we see that $\tan 2$ and $\tan 3$ are negative, while $\tan 1$ is positive. Furthermore, we have $\tan 2 < \tan 3$. The answer is **c**.

14. The total number of times the six mathematicians will read theses is $6 \cdot 102 = 612$ times. Each thesis is counted in this total four times, so the number of theses is $612/4 = 153$. The answer is **a**.

15. Suppose that we color each of the six letters a different color, in order to distinguish them. Then the number of permutations of the six colored letters is $6! = 720$. Each of these permutations corresponds to $2! \cdot 3! = 2 \cdot 6 = 12$ banana strings, which are obtained from it by ignoring the colors. (The point is that we can permute the two Ns and three As in 12 ways for each fixed colored permutation, and these 12 permutations correspond to the same (uncolored) banana string.) Hence the total number of banana strings is $720/12 = 60$, and the answer is **b**.

16. The function f is piecewise linear, meaning that it is linear on each of the 5 intervals $(-\infty, 1]$, $[1, 2]$, $[2, 3]$, $[3, 4]$, and $[4, +\infty)$. We compute the value of $f(x)$ on each of these intervals as $10-4x$, $8 - 2x$, 4, $2x - 2$, and $4x - 10$, respectively. We conclude that the answer is **b**.

17. Suppose that the vertices of the angles of x, $2x$, and $3x$ degrees are M, N, and A, respectively, and that the inscribed quadrilateral has vertices $ABCD$, listed clockwise so that C is opposite the vertex A. Since $ABCD$ is inscribed in a circle, the angle $\angle BCD = 180° - 3x$, and thus $\angle DCM = 3x$. In addition, $\angle NDM$ is an exterior angle of the triangle AND, hence $\angle NDM = 2x + 3x = 5x$. As the sum of the interior angles of triangle CDM is 180°, we deduce that $3x + 5x + x = 180°$, or $x = 20°$. The answer is **d**.

18. The given condition on the natural number n is that there exist integers k and m greater than 4 such that

$$n = (k-4) + (k-3) + \cdots + (k+3) + (k+4) = (m-4) + (m-3) + \cdots + (m+4) + (m+5)$$

or equivalently, $n = 9k = 10m + 5 = 5(2m+1)$. It follows that 9 divides $2m + 1$, and since $m > 4$ the least possible value of m is 13. Therefore $k = 15$, $n = 9 \cdot 15 = 135$, and the answer is **d**.

19. Either Charlie or David must be lying. If Charlie told the truth, then both Adam and Bob are lying, and hence so is David, which is impossible. Therefore Charlie is lying. If David is truthful, then either Adam or Bob is lying, and hence Ethan told the truth (as there are only two liers among the five). But then David must be lying, which is a contradiction. We deduce that David lied, and therefore that both Adam and Bob told the truth. It follows that Charlie baked the cake, and the answer is **c**.

20. We are given that $r^2 = r + 10$. Now we use this to compute

$$(r+1)(r+2)(r-4) = (r^2 + 3r + 2)(r-4) = (4r+12)(r-4) = 4(r^2 - r - 12) = -8.$$

The answer is **a**.

21. We have $f(x) = 1 + (-x) + (-x)^2 + \cdots + (-x)^{10} = (1 + x^{11})/(1 + x)$, using a well known identity. Now $g(y) = f(y + 1)$, so $a_0 + a_1 + \cdots + a_{10} = g(1) = f(2) = (1 + 2^{11})/3 = 2049/3 = 683$. The answer is **d**.

22. If the prime power decomposition of 100! is $2^a 3^b 5^c \cdots$, then we need to determine a and b. To find a, observe that there are $100/2 = 50$ even numbers among the first 100 positive integers, and among these there are $100/4 = 25$ multiples of 4, $\lfloor 100/8 \rfloor = 12$ multiples of 8, $\lfloor 100/16 \rfloor = 6$ multiples of 16, $\lfloor 100/32 \rfloor = 3$ multiples of 32, and $\lfloor 100/64 \rfloor = 1$ multiple of 64. It follows that $a = 50 + 25 + 12 + 6 + 3 + 1 = 97$. We similarly compute that $b = \lfloor 100/3 \rfloor + \lfloor 100/9 \rfloor + \lfloor 100/27 \rfloor + \lfloor 100/81 \rfloor = 33 + 11 + 3 + 1 = 48$. Finally, we see that

$$\frac{100!}{18^n} = \frac{2^{97} 3^{48} 5^c \cdots}{2^n 3^{2n}} = 2^{97-n} 3^{48-2n} 5^c \cdots$$

This fails to be an integer when $48 - 2n < 0$, or $n > 24$. The answer is **b**.

23. Suppose that the centers of the four circles are the points E, F, G, and H, so that A, B, C, and D lie on the segments EF, FG, GH, and HE, respectively. It follows that $|EF| + |GH| = |FG| + |HE|$, and therefore that there is a circle $\mathcal{C}$ tangent to the four sides of quadrilateral $EFGH$. As $EFGH$ is a circumscribed quadrilateral, we deduce that the four interior bisectors of its angles meet in a single point O, which is the center of the inscribed circle $\mathcal{C}$. Since the points on the bisector of any angle are equidistant from its sides, we deduce that $|OA| = |OB| = |OC| = |OD|$, and hence that O is the center of a circle which passes through A, B, C, and D. The answer is **a**.

24. We compute that

$$\cos(80°) + \cos(40°) = 2\cos\frac{80° + 40°}{2}\cos\frac{80° - 40°}{2} = 2\cos(60°)\cos(20°) = \cos(20°).$$

Therefore, $\cos(20°) - \cos(40°) + \cos(60°) - \cos(80°) = \cos(60°) = 1/2$. The answer is **e**.

25. The total number of games played is 10, hence the players shared 10 points among themselves at the end of the tournament. The player A in first place must have lost to the second placed player B, so player A scored at most 3 points. But then the final scores of the five players must have been 3, 2.5, 2, 1.5, and 1, since these five numbers sum to 10 and all scores are distinct integers less than or equal to 3. We deduce that player A won 3 games and lost 1 to player B, while the remaining three games of B where all draws. Finally, players C, D, and E all lost to A and drew with B, and the only way to get final scores of 2, 1.5, and 1 as listed is if they had exactly two more draws amongst themselves. Therefore the answer is **d**.