THE 35$^{\text{th}}$ ANNUAL (2013) UNIVERSITY OF MARYLAND

HIGH SCHOOL MATHEMATICS COMPETITION

PART I MULTIPLE CHOICE

For each of the following 25 questions, carefully blacken the appropriate box on the answer sheet with a #2 pencil. Do not fold, bend or write stray marks on either side of the answer sheet. Each correct answer is worth 4 points. **Two points are deducted for each incorrect answer.** *Zero points are given if no box, or more than one box, is marked. Note that wild guessing is apt to lower your score. When the exam is over, give your answer sheet to the proctor. You may keep your copy of the questions.*

NO CALCULATORS

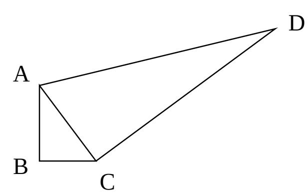
75 MINUTES

1. Which of the following numbers has the largest absolute value?
   a. -1.1 b. $\frac{100}{101}$ c. 1.01 d. $-\frac{1}{2}$ e. $-\frac{2}{1}$

2. If we fold a rectangular rug in half along its shorter midline, the two sides of the resulting rectangle are in the same ratio as the original rectangle. What is the ratio of the length of the rug to its width?
   a. $\sqrt{2}$ b. 2 c. $\sqrt{3}$ d. 3 e. 4

3. An alchemist knows that one jar equals one bottle plus one glass, two jars equal seven glasses, and one bottle equals one cup plus two glasses. How many cups are there in a bottle?
   a. 2 b. 3 c. 4 d. 5 e. 6

4. In quadrilateral $ABCD$, angle $\angle ABC$ and angle $\angle ACD$ are right angles. Moreover, the lengths $|AB| = 4$, $|BC| = 3$, and $|CD| = 12$. How long is the side AD?



   a. 12 b. 12.5 c. 13 d. 13.5 e. 14

5. A student takes a test where each problem counts for a certain number of points and the final score is the average of these points. A computer checks the answers, so the student knows the score for each problem immediately after he completes it. The student realizes that if he gets 97 points for the last problem, then his final score will be 91; if he gets 73 points for the last problem, then his final score will be 88. How many problems are there on the test?
   a. 6 b. 7 c. 8 d. 9 e. 10

6. A number is *useful* if it is divisible by the sum of its digits. How many of the following numbers are useful?

   11, 111, 1111, 11111, 111111, 1111111, 11111111, 111111111

   a. 1 b. 2 c. 3 d. 4 e. 5

7. If θ is an angle such that $\sin\theta = \sqrt{3}/3$, then $8\sin^2\theta + 7\cos^2\theta$ equals
   a. 5 b. $4\sqrt{3}$ c. $5\sqrt{3}$ d. $10\sqrt{3}/3$ e. $22/3$

8. The perimeter of square of area 1 is equal to the circumference of a circle C. Which of the following numbers is closest to the area of C?
   a. 1 b. 1.1 c. 1.2 d. 1.3 e. 1.4

9. Solve the following equation.

$$2\log_{10}(x) + \log_{10}(x+4) = \log_{10}(4) + \log_{10}(x) + \log_{10}(x+1).$$

   a. $x = 1$ b. $x = 2$ c. $x = \sqrt{10}$ d. $x = 3 + \sqrt{10}$ e. $x = 5$

10. Let L be the line that is tangent to the circle $(x-7)^2 + (y-4)^2 = 25$ at the point $(3, 7)$. The line L intersects the y-axis at the point $(0, b)$. What is b?
    a. -2 b. 0 c. 3 d. 5 e. 9

11. A swimming pool has an input pump for filling the pool and an output pump for emptying the pool. The input pump can fill the pool in 3 hours, and the output pump can drain the pool in 5 hours. As you go to bed, the pool is full, but a neighbor's kid turns on the output pump. At midnight, you awake to find the pool half empty. Immediately, you turn on the input pump, but you are sleepy and forget to turn off the output pump. At what time will the pool become full?
    a. 1:30 am b. 2:45 am c. 3:30 am d. 3:45 am e. 4:30 am

12. We know that each one of the displayed five cards (shown face up) has a number on one side and a letter on the other side.

    $\boxed{\text{S}}\ \boxed{3}\ \boxed{\text{H}}\ \boxed{3}\ \boxed{8}$

    Consider the assertion: 'If a card has an S on one side, then it has a 3 on the other side.' Let m be the least number of cards that you need to turn over in order to prove that the assertion is true. Determine m.
    a. 1 b. 2 c. 3 d. 4 e. 5

13. Wile E. Coyote and Road Runner have a 100-mile race. Road Runner runs 10 miles per hour faster than Wile E. Coyote and finishes 1 hour ahead of Wile E. Coyote. How fast does Road Runner run (in miles per hour)?
    a. $5 + 5\sqrt{41}$ b. 37 c. $\sqrt{110}$ d. 120 e. $30 + \sqrt{37}$

14. For how many positive integers n is $2^{2^{2^n}} < 10^{10^n}$?
    a. 1 b. 2 c. 3 d. 4 e. An infinite number of n

15. Consider the set $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. For every subset A of S, John computes the sum of all elements in A and writes the result on the blackboard. Mary then computes the sum of all the numbers that John wrote on the board. The final result is equal to
    a. 11385 b. 11430 c. 11475 d. 11500 e. 11520

16. The integers from 2 to 1000 are written on the blackboard. The students in school play the following game. Each student in turn picks a number on the blackboard and erases it together with all of its multiples. The game ends when only primes are left written on the board. What is the smallest number of students that need to play before the game ends?
    a. 11 b. 31 c. 51 d. 71 e. 91

17. The cubic polynomial $ax^3 + bx + c$ is divisible by the quadratic polynomial $x^2 + dx + 1$. Which of the following must be true?
a. $b^2 \geq 4ac$ b. $a^2 - c^2 = ab$ c. $a^2 - b^2 = c^2$ d. $a^2 + c^2 = 2b^2$ e. $a^2 + c^2 = ab$

18. Consider an arbitrary convex quadrilateral $ABCD$. For which of the following points P is the sum of the distances from P to the vertices of $ABCD$ the smallest?
a. The center of mass of $ABCD$ b. One of the vertices of $ABCD$ c. A point outside of $ABCD$ d. The intersection point of the diagonals of $ABCD$ e. The intersection point of the two line segments connecting the midpoints of opposite sides of $ABCD$

19. A certain integer n satisfies $n^5 = 12608989261857$. What is the leading digit of n?
a. 1 b. 2 c. 3 d. 4 e. 5

20. In the sum computation below, each letter represents a distinct digit. Which of the following digits is represented by U?

$$\begin{array}{rccccc} & \text{S} & \text{U} & \text{A} & \text{V} & \text{E} \\ & & \text{S} & \text{A} & \text{G} & \text{E} \\ + & & \text{S} & \text{A} & \text{G} & \text{E} \\ \hline & 4 & 6 & 9 & 3 & 3 \end{array}$$

a. 0 b. 5 c. 7 d. 8 e. 9

21. Suppose that a and b are real numbers such that both $a + b$ and ab are integers. Which of the following *cannot* occur?
a. a and b are both negative b. $a - b$ is not an integer c. a and b are rational but not integers d. a and b are both irrational e. a and b are both irrational and positive

22. Let $ABCD$ be a tetrahedron. A plane is called *balanced* if all four vertices of $ABCD$ have the same distance from it. How many balanced planes are there?
a. 1 b. 4 c. at most 6 d. 7 e. 10

23. If the relation

$$c^4 - 2(a^2 + b^2)c^2 + a^4 + a^2b^2 + b^4 = 0$$

holds among the side lengths a, b, and c of triangle ABC, then angle $\angle ACB$ is equal to
a. $30°$ b. $60°$ c. $120°$ d. $30°$ or $120°$ e. $60°$ or $120°$

24. The *first quadrant* is the set of all points in the Cartesian plane whose coordinates are both positive. We join 20 distinct random points on the positive x-axis to 20 distinct random points on the positive y-axis with lines. Suppose that no three of these 400 lines pass through the same point in the first quadrant. Find the total number of intersection points of these lines in the first quadrant.
a. 36100 b. 38000 c. 40000 d. 42000 e. 79800

25. A group of 2013 students $S_1, ..., S_{2013}$ sit in this order around a circle, going clockwise. Starting from student S_1 with the number 1, and going clockwise, they consecutively count the numbers 1, 2, and 3, and repeat. Each student that counts 2 or 3 as they do this must leave the circle, and they continue until only one student S_k remains. Determine k.
a. 1927 b. 1972 c. 1297 d. 1729 e. 1792