

THE 35th ANNUAL (2013) UNIVERSITY OF MARYLAND
HIGH SCHOOL MATHEMATICS COMPETITION
PART I SOLUTIONS

1. The absolute values satisfy $\frac{1}{2} < \frac{100}{101} < 1.01 < 1.1 < \frac{2}{1}$. The answer is **e**.
2. If the short side of the rug is x and the long side y , then after folding the roles of x and y are replaced by $y/2$ and x . We know that $x/y = (y/2)/x$, so $y^2 = 2x^2$, which gives $y/x = \sqrt{2}$. The answer is **a**.
3. Let the volumes of a jar, bottle, glass, and cup be given by j, b, g , and c , respectively. We are told that $j = b + g$, $2j = 7g$, and $b = c + 2g$. The first two equations imply that $7g = 2(b + g) = 2b + 2g$ and hence $5g = 2b$. Multiplying the last equation by 5 and substituting $2b$ for $5g$ gives $5b = 5c + 10g = 5c + 4b$, or $b = 5c$. The answer is **d**.
4. Using the Pythagorean theorem in the right triangles ABC and ACD gives $|AC|^2 = 3^2 + 4^2 = 25$ and $|AD|^2 = |AC|^2 + 12^2 = 25 + 144 = 169$. Therefore, $|AD| = \sqrt{169} = 13$. The answer is **c**.
5. Suppose that there are n problems on the test, and that before doing the last problem, the student's score is d points. We are given that $(d + 97)/n = 91$ and $(d + 73)/n = 88$. Therefore $3n = 91n - 88n = (d + 97) - (d + 73) = 24$, and hence $n = 8$. The answer is **c**.
6. Since all the numbers are odd, the useful numbers among the eight given must have an odd number of digits. 11111 is not divisible by 5 since it does not end in 5, and one checks that 1111111 is not divisible by 7. This leaves $111 = 3 \cdot 37$ and $111111111 = 9 \cdot 12345679$ as the only two useful numbers on the list. The answer is **b**.
7. We have $8\sin^2\theta + 7\cos^2\theta = \sin^2\theta + 7(\sin^2\theta + \cos^2\theta) = 1/3 + 7 = 22/7$. The answer is **e**.
8. If r is the radius of C , then its circumference equals $2\pi r$, and this is equal to 4 (since the square has four sides, each of length 1). Therefore $r = \frac{2}{\pi}$ and the area of C is $\pi r^2 = \frac{4}{\pi}$. Since π is approximated by 3.14 to two decimal places, we see that $1.26\dots = \frac{4}{3.15} < \frac{4}{\pi} < \frac{4}{3.13} = 1.27\dots$. The answer is **d**.
9. Since $\log_{10}(x)$ is defined, we have $x > 0$. Observe that $2\log_{10}(x) + \log_{10}(x+4) = \log_{10}(x^2(x+4))$ and $\log_{10}(4) + \log_{10}(x) + \log_{10}(x+1) = \log_{10}(4x(x+1))$. Therefore, $x^2(x+4) = 4x(x+1)$, which implies $x^2 + 4x = 4x + 4$ and hence $x^2 = 4$. Since $x > 0$, we obtain $x = 2$. The answer is **b**.
10. Let $A(7, 4)$ be the center of the given circle and $B(3, 7)$ be the given point on it. The line through A and B has slope $(7 - 4)/(3 - 7) = -3/4$, and hence L has slope $4/3$. The equation of the line L is therefore $y - 7 = \frac{4}{3}(x - 3)$. Setting $(x, y) = (0, b)$ gives $b = 7 - 4 = 3$. The answer is **c**.
11. Let V be the volume of half of the pool. The input pump can fill V gallons of water in $3/2$ hours, so each hour it fills $2/3$ of V . The output pump can drain half the pool in $5/2$ hours, so each hour it drains $2/5$ of V . The combined action of the two pumps fills $2/3 - 2/5 = 4/15$ of V each hour. Therefore, the pool will become full in $15/4 = 3 + 3/4$ hours, at time 3:45 am. The answer is **d**.

12. Clearly we need to turn over the first card with the S, to see if it has a 3 on the other side, and the fifth card with the 8, in case it has an S on the other side. The H card has a number on the other side, so we need not turn that over. Finally, it does not matter what letter(s) are on the reverse sides of the two cards with a 3, so they need not be turned over either. The answer is **b**.
13. Let Wile E. Coyote and Road Runner run at v_c and v_r miles per hour, respectively. Then $v_r - v_c = 10$ and $100/v_c - 100/v_r = 1$, hence $v_c v_r = 100(v_r - v_c) = 1000$. Since $v_c = v_r - 10$, we get the quadratic equation $v_r^2 - 10v_r - 1000 = 0$ for v_r . Solving this gives the single positive root $v_r = (10 + \sqrt{4100})/2 = 5 + 5\sqrt{41}$. The answer is **a**.
14. We have $16 < 10^{10}$, $2^{16} < 10^{100}$, and $2^{2^8} = 2^{256} < 10^{1000}$, so the inequality holds for $n \in \{1, 2, 3\}$. However, we have

$$2^{2^{2^4}} = 2^{2^{16}} = (2^4)^{2^{14}} > 10^{2^{14}} > 10^{10000}$$

and it follows that the inequality fails for all $n > 3$. The answer is **c**.

15. The set of all subsets of S may be partitioned into pairs, with each pair consisting of a given subset and its complement. Clearly the sum of all the elements in any pair is $1 + 2 + \dots + 9 = 45$. Since the set S has exactly 2^9 subsets, the total numbers of such pairs will be $2^9/2 = 2^8$. Therefore the end result of Mary's computation will be $45 \cdot 2^8 = 11520$. The answer is **e**.
16. If the students want to minimize the total number of turns to be played, then each student should pick a prime among the numbers that remain on the blackboard (unless there are no such). We claim that choosing the 11 primes 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31 whose square is less than 1000 will suffice to end the game. This follows because any composite number $n \leq 1000$ must have a prime factor which is at most 31. Indeed, if n has two prime factors p and q which are both at least 37, then $n \geq pq \geq 37^2 > 1000$, which is a contradiction. The answer is **a**.
17. The assumption is equivalent to the existence of an e such that $ax^3 + bx + c = (ax + e)(x^2 + dx + 1)$. Expanding the right hand side into $ax^3 + (ad + e)x^2 + (a + ed)x + e$ and equating the like powers of x gives the three equations $ad + e = 0$, $a + ed = b$, and $e = c$. We deduce that $ad = -c$ and $a + cd = b$, and hence that $ab = a(a + cd) = a^2 + c(ad) = a^2 - c^2$. The answer is **b**.
18. The shortest path between any two points is a straight line. This implies that $|PA| + |PC|$ is smallest when P lies on the diagonal AC , and similarly $|PB| + |PD|$ is smallest when P lies on the diagonal BD . Hence $|PA| + |PC| + |PB| + |PD|$ is least when P is the point where the two diagonals intersect. The answer is **d**.

19. We have

$$4^5 \cdot 10^{10} < 1260 \cdot 10^{10} < n^5 < 1270 \cdot 10^{10} < 5^5 \cdot 10^{10}.$$

Taking 5-th roots gives $400 < n < 500$. The answer is **d**.

20. Notice first that $E = 1$. It is easy to see that S cannot be 2, so S is either 3 or 4. But if $S = 4$ then in order to get a 6 in the second column, U must be large enough to induce a carry over to the first column, which gives a contradiction. Therefore S must equal 3, and U is either 8 or 9. But $U = 8$ can only happen if $A = 9$, which leads to $G = 7$ and $V = 9$, a contradiction. We deduce that $U = 9$, and the answer is **e**. In fact the sum must be the one shown below.

$$\begin{array}{r} 3 \ 9 \ 6 \ 5 \ 1 \\ \ 3 \ 6 \ 4 \ 1 \\ + \ 3 \ 6 \ 4 \ 1 \\ \hline 4 \ 6 \ 9 \ 3 \ 3 \end{array}$$

21. Clearly (a) can occur, e.g. let $a = b = -1$. Setting $a = 2 + \sqrt{2}$ and $b = 2 - \sqrt{2}$ shows that (b), (d), and (e) are also possible. The answer therefore must be **c**. In fact, if $a + b = s$ and $ab = p$, then a and b are roots of the quadratic equation $x^2 - sx + p = 0$, with p and s both integers. It is a theorem that if such an equation has rational roots, then the two roots must be integers.
22. A balanced plane cannot have all four vertices of $ABCD$ on the same side, for otherwise they would all have to lie on the same plane. There are four balanced planes that have three vertices on one side of them, and one on the other side; these are parallel to the four faces of the tetrahedron. There are three balanced planes that have two vertices on either side. To see this last claim, say that A, B and C, D are the two pairs of vertices that are separated by the balanced plane Π . Then AB and CD are two skew lines in space. These lines determine a unique pair Π_1 and Π_2 of parallel planes, with AB lying on Π_1 and CD lying on Π_2 . Then Π is the plane parallel to Π_1 and Π_2 which lies halfway between them. A similar construction works for the pairs of skew lines AC, BD and AD, BC . We conclude that there are seven balanced planes, so the answer is **d**.
23. The law of cosines in triangle ABC gives $2ab \cos(\angle ACB) = a^2 + b^2 - c^2$. Squaring this, we obtain $4a^2b^2 \cos^2(\angle ACB) = (a^2 + b^2 - c^2)^2 = a^4 + b^4 + 2a^2b^2 + c^4 - 2(a^2 + b^2)c^2$. Using the given relation, we see that $4a^2b^2 \cos^2(\angle ACB) = a^2b^2$, or $\cos(\angle ACB) = \pm 1/2$. This in turn implies that $\angle ACB$ is either 60° or 120° . The answer is **e**.
24. Let A and B be two of the 20 points on the positive x -axis, and C and D be two of the 20 points on the positive y -axis. Then there is a unique point in the first quadrant obtain by intersecting the 4 lines joining A and B to C and D (the intersection of the two diagonals of the quadrilateral with vertices A, B, C , and D). We thus see that each intersection point in the first quadrant corresponds to two pairs of points, one pair from the 20 points on the x -axis and one pair from the 20 points on the y -axis. There are $\binom{20}{2}$ ways to choose each of these pairs, so we conclude that there are $\binom{20}{2}^2 = 190^2 = 36100$ intersection points in total. The answer is **a**.
25. If there are 3^n students sitting around the circle, then after the first round of counting there remain 3^{n-1} students and student S_1 will count the same number 1 in the second round. So after n rounds, S_1 will be the last student remaining. Now suppose that there are 2013 students. Since $3^6 = 729 < 2013 < 3^7 = 2187$, we can reduce our problem to the previous easy case when there are 3^6 students left; then student S_k will be the student who will count 1 first among these 3^6 students. We need to remove $2013 - 729 = 1284 = 2 \cdot 642$ students. These correspond to 642 groups of three students (with two left out from each group). So we need $642 \cdot 3 = 1926$ students sitting before S_k , and hence $k = 1927$. The answer is **a**.