THE 36th ANNUAL (2014) UNIVERSITY OF MARYLAND

HIGH SCHOOL MATHEMATICS COMPETITION

PART I SOLUTIONS

1. We have $3^{(2^3)} > (3^2)^3 > (3^2) \cdot 3 > 3 \cdot 2^3 > 3 \cdot 2 \cdot 3$. Since $n \mapsto 2^n$ is an increasing function of n, we deduce that $2^{3^{(2^3)}} > 2^{(3^2)^3} > 2^{(3^2) \cdot 3} > 2^{3 \cdot 2^3} > 2^{3 \cdot 2 \cdot 3}$. The answer is **a**.

2. There are $365 \cdot 24 \cdot 60 \cdot 60 = 31,536,000$ seconds in the year 2014, so Galois will donate \$315, 360. The answer is **c**.

3. Let d be the length of the path up the hill, and t the total time for their trip up the hill. Then $d = 2t$. Since the time for their trip down the hill was $\frac{1}{2} - t$, we also get $d = 10(\frac{1}{2} - t) = 5 - 10t$. We deduce that $2t = 5 - 10t$, so $t = \frac{5}{12}$ and $d = \frac{5}{6}$. The answer is **c**.

4. Substitute $x = y + 1$ into the equation of the hyperbola to obtain $(y-2)^2 = 1$. This last equation has the two solutions $y = 1$ and $y = 3$. The points A and B are therefore $(2, 1)$ and $(4, 3)$, and the distance between them is $\sqrt{(4-2)^2 + (3-1)^2} = \sqrt{8} = 2\sqrt{2}$. The answer is **a**.

5. Since $a_1 + a_2 + a_3 = a_2 + a_3 + a_4$, we deduce that $a_4 = a_1 = 7$. We similarly have $a_4 + a_5 + a_6 = a_5 + a_6 + a_7$, hence $a_7 = a_4 = 7$. Now $25 = a_6 + a_7 + a_8 = a_6 + 16$, so $a_6 = 9$. Working our way back in threes, we deduce that the sequence is $7, 9, 9, 7, 9, 9, 7, 9$. The answer is **d**.

6. If the system has no solutions, then the 2×2 determinant of coefficients $36n - 12p$ must equal zero. This gives $p = 3n$, and since p is prime we deduce that $n = 1$ and $p = 3$. The answer is **a**.

7. We first simplify by dividing all three numbers by 20 and equivalently comparing $A' = 10\pi$, $B' = (1.5)^{3/2}(266)^{1/2}$, and $C' = 30$. Since $\pi = 3.14\ldots$ we clearly have $C' < A'$ and hence $C < A$. We also have

$$B' = (1.5)^{3/2}(266)^{1/2} = \left(\frac{3}{2}\right)^{3/2}\sqrt{266} = \frac{3}{2}\sqrt{\frac{3}{2}}\sqrt{266} = \frac{3}{2}\sqrt{399} < \frac{3}{2}\sqrt{400} = \frac{3}{2}(20) = 30 = C'.$$

We conclude that $B' < C'$, and hence $B < C < A$. The answer is **e**.

8. Notice first that $x > 1$ implies that $\log_{10}(x) > 0$. Squaring both sides of the given equation, we obtain $\log_{10}^2(\sqrt{x}) = \log_{10}(x)$, hence $\frac{1}{4}\log_{10}^2(x) = \log_{10}(x)$. Cancelling $\log_{10}(x)$ from both sides gives $\log_{10}(x) = 4$, or $x = 10^4 = 10000$. The answer is **d**.

9. The required number is the least common multiple of $9 = 3^2$, $10 = 2 \cdot 5$, 11, and $12 = 2^2 \cdot 3$, and therefore is $2^2 \cdot 3^2 \cdot 5 \cdot 11 = 1980$. The answer is **b**.

10. Since $\tan(\theta) \cdot \tan(90° - \theta) = 1$ for any angle θ, we deduce that

$$\tan 10° \cdots \tan 80° = (\tan 10° \tan 80°) \cdot (\tan 20° \tan 70°) \cdot (\tan 30° \tan 60°) \cdot (\tan 40° \tan 50°) = 1.$$

The answer is **a**.

11. Let g be the fraction of gold weight and d the fraction of diamond weight contained in the 100 pound sack that Ali Baba carries away. Then we have $g + d = 1$ and the assumptions on the weight of gold and diamonds give $200g + 40d = 100$. Solving the system of two equations, we obtain $g = 3/8$ and $d = 5/8$, so there are $200 \cdot (3/8) = 75$ pounds of gold and $40 \cdot (5/8) = 25$ pounds of diamonds in the sack. The answer is therefore $75 \cdot 20 + 25 \cdot 60 = 3000$, so **c**.

12. Let P be the top vertex, $ABCD$ be the square base of the pyramid, and Q and M be the orthogonal projections of P onto the plane $ABCD$ and the side AB, respectively. Observe that PM is an altitude of the equilateral triangle PAB, hence $|PM| = \sqrt{3}/2$. We have $|QM| = 1/2$, and the Pythagorean theorem applied to the right triangle PQM gives the height h of the pyramid as $h = |PQ| = \sqrt{|PM|^2 - |QM|^2} = \sqrt{2}/2$. Since the area of the base $ABCD$ equals 1, we conclude that the volume of the pyramid is $(1/3) \cdot 1 \cdot \sqrt{2}/2 = \sqrt{2}/6$. The answer is **c**.

13. Notice that every move reduces the total number of pieces by one. Since we begin with 400 pieces, any assembly of the puzzle must use exactly 399 moves. The answer is **a**.

14. Let d_n be the last digit of the number a_n. Then the sequence $\{d_n\}$ begins with

$$1, 3, 4, 7, 1, 8, 9, 7, 6, 3, 9, 2, 1, 3, 4, \ldots.$$

Notice that $\{d_n\}$ is a periodic sequence with period 12. Since $2014 = 167 \cdot 12 + 10$, we deduce that the the required last digit of a_{2014} is equal to $d_{10} = 3$. The answer is **b**.

15. Use the fundamental theorem of arithmetic to write $P = 2^a 5^b m$, where m is a number relatively prime to 10. Since no number in the array is a multiple of 25, we find b simply by counting the number of entries which are multiples of 5. These fall into four diagonals, with 4, 9, 6, and 1 entry, respectively, and thus the exponent b satisfies $b = 4+9+6+1 = 20$. It is easy to see that $a \geq b$, so we have $P = 10^{20} n$, where n is not a multiple of 5. This implies that there are exactly 20 zeroes at the end of the number P. The answer is **d**.

16. The region S is a square centered at the point $(3, 6)$ and whose vertices are the points $(13, 6)$, $(-7, 6)$, $(3, 16)$, $(3, -4)$. This square has a side of length $10\sqrt{2}$, and therefore area $(10\sqrt{2})^2 = 200$. The answer is **d**.

17. The five cards can fill the three envelopes in two ways: (a) 3 cards go in one envelope and 1 card in each of the remaining two, or (b) 2 cards go in each of 2 envelopes, and 1 in the remaining one. In case (a), we can select the envelope with 3 cards in 3 ways, and there are $\binom{5}{3} = 10$ choices for the cards that go in it. For each of these choices, there are 2 possible placements for the remaining two cards. This gives a total of $10 \cdot 3 \cdot 2 = 60$ possibilities. In case (b), there are 3 choices for the singleton envelope, and 5 distinct cards to place in it. For each of these 15 choices, there are 6 ways to fill the remaining 2 envelopes with 2 cards in each. This gives a total of $15 \cdot 6 = 90$ possibilities. We obtain a grand total of $60 + 90 = 150$ ways of distributing the 5 cards into the 3 envelopes. The answer is **e**.

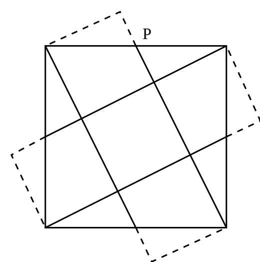
18. The equation $ab = 10(a+b)$ can be written as $1/a + 1/b = 1/10$, and since $b \leq a$, we deduce that $10 < b \leq 20$. Hence there are 10 possibilities for b, and for each of these, we can compute a using the equation $a = 10b/(b-10)$. This gives a total of 5 pairs (a, b) which satisfy the conditions, namely $(110, 11)$, $(60, 12)$, $(35, 14)$, $(30, 15)$, and $(20, 20)$. The answer is **d**.

19. Imagine that when two ants meet, they pass through each other instead of reversing directions. These two points of view produce the same picture on the ruler, but the former is easier to analyze, for then each ant will simply crawl to the end of the ruler and fall off. The total distance crawled by the ants is therefore $(11 + 10 + 9 + 8 + 7) + (6 + 7 + 8 + 9 + 10 + 11) = 45 + 51 = 96$. The answer is **b**.

20. Set $a = \sqrt{4+\sqrt{15}}$. Then $1/a = \sqrt{4-\sqrt{15}}$, since $(4+\sqrt{15})(4-\sqrt{15}) = 1$. If $y = a^x$, then we have $y + 1/y = 8$, or equivalently $y^2 - 8y + 1 = 0$. This gives the answers $y = 4+\sqrt{15} = a^2$ and $y = 4-\sqrt{15} = 1/a^2 = a^{-2}$, and therefore $x = 2$ and $x = -2$. The answer is **c**.

21. Suppose that there are a tunnels between B and C, b tunnels between A and C, and c tunnels between A and B. The conditions of the problem give $ab + c = 89$ and $ac + b = 82$, and we are asked to determine $N := bc + a$. Subtracting the second equation from the first gives $(a-1)(b-c) = 7$. Since 7 is a prime number, we deduce that $a - 1 \in \{1, 7\}$, and so $a = 2$ or $a = 8$. If $a = 2$ then $b = c + 7$ and $ac + b = 3c + 7 = 82$, so $c = 25$, $b = 32$, and $N = 802$. If $a = 8$ then $b = c + 1$ and we similarly obtain $c = 9$, $b = 10$, and $N = 98$. Since the number we seek is greater than 99, we deduce that $N = 802$ is correct. The answer is **e**.

22. Let T be one of the four small triangles at the corners of the original figure. Then T has a vertex P which is the midpoint of a side of the large square. Rotate the triangle T 180° around the center of rotation P. When this is done to each of the four small triangles, we obtain five equal squares in the shape of a cross, as shown in the figure below. The total area of these 5 squares is equal to the area of the large square. Therefore, the area of the smaller square in the original figure is $1/5$. The answer is **b**.



23. The set A contains at most 2 elements of $\{1, 2, 3, 5, 7, 11, 13\}$, so A can have at most $16 - 5 = 11$ elements. On the other hand, the set $A = \{2, 4, 6, 8, 10, 12, 14, 16, 3, 9, 15\}$ has 11 elements and satisfies the required property. Indeed, if T is any subset of A containing three elements, two of them must either be even or must lie in $\{3, 9, 15\}$, and in both cases they will have a common factor (2 or 3, respectively) which is larger than 1. The answer is **c**.

24. Number the vertices of the octagon from 1 to 8 in clockwise order. The symbol $[a, b, c, d]$ will denote a coloring whose red vertices are numbered by a, b, c, and d, with $a < b < c < d$. If X is a coloring with no adjacent red vertices, then it is the same as $[1, 3, 5, 7]$. Otherwise, we may assume, by rotating the polygon if necessary, that $a = 1$, $b = 2$, and $d \leq 7$. With $c = 3$ we obtain the 4 distinct colorings $[1, 2, 3, m]$ with $4 \leq m \leq 7$. With $c = 4$ we obtain the 3 distinct colorings $[1, 2, 4, n]$ with $5 \leq n \leq 7$. With $c = 5$ we obtain the 2 distinct colorings $[1, 2, 5, 6]$ and $[1, 2, 5, 7]$. This exhausts all possibilities, since $[1, 2, 6, 7]$ coincides with $[1, 2, 4, 5]$. The total number of colorings is thus $1 + 4 + 3 + 2 = 10$. The answer is **b**.

25. Set $x := |AC|$ and $\theta = \angle CDB$. The Pythagorean theorem gives $|BC| = \sqrt{x^2 - 1}$. The law of signs in triangle BCD gives $\sin(\theta)/\sqrt{x^2-1} = \sin(30°)/1 = 1/2$, and the law of signs in triangle ABD gives $\sin(\theta)/1 = \sin(120°)/(1 + x) = \sqrt{3}/(2(1 + x))$. Eliminating $\sin(\theta)$ from these two equations, we obtain $(1 + x)\sqrt{x^2 - 1} = \sqrt{3}$. Squaring gives $(1 + x)^2(x^2 - 1) = 3$, which simplifies to $x^4 + 2x^3 - 2x - 4 = 0$. Since $x^4 + 2x^3 - 2x - 4 = (x + 2)(x^3 - 2)$ and $x > 0$, we see that the unique solution is $x = \sqrt[3]{2}$. The answer is **a**.