THE 37th ANNUAL (2015) UNIVERSITY OF MARYLAND
HIGH SCHOOL MATHEMATICS COMPETITION
PART I SOLUTIONS

1. The required number is the remainder of the division of 2015 by 12. Since $2015 = 167 \cdot 12 + 11$, this remainder is 11. The answer is **(e)**.

2. There are $24 \cdot 60 \cdot 60 = 86400$ seconds in every day. Since $1000000/86400 = 11.57\ldots$, the count will take 11.57 days. The answer is **(d)**.

3. The speed of the westbound train relative to the engineer is $60 + 30 = 90$ miles per hour, or $44 \cdot 3 = 132$ feet per second. Since it takes 30 seconds for the westbound train to pass by his window, the length of the westbound train is $132 \cdot 30 = 3960$ feet. The answer is **(c)**.

4. We have $3/3 > 3/4$, $2/4 < 2/3$, while $3/5 < 2/3 < 3/4 < 4/5$ and $2/3 = 4/6 < 3/4 < 5/6$. Finally, since $2/3 < 5/7 < 3/4$, the answer is **(d)**.

5. Let x (respectively, y) be the amount of calories in one cookie (respectively, in one candy bar). The assumptions of the problem give the two equations $30x + 40y = 8000$ and $20x + 50y = 7900$. Simplify these to get $3x + 4y = 800$ and $2x + 5y = 790$. Subtracting the second equation from the first gives $x - y = 10$, so $y = x - 10$. Substitute this value of y into the first equation to obtain $7x - 40 = 800$, and hence $x = 120$. The answer is **(b)**.

6. Taking square roots, we have $x^2 > y^2$ if and only if $\sqrt{x^2} > \sqrt{y^2}$. Since $\sqrt{x^2} = |x|$, the answer is **(e)**.

7. The required number of seconds t satisfies the equation $9000-16t^2 = 8880-8t$, which simplifies to the quadratic equation $2t^2-t-15 = 0$. The positive root of this equation is $t = (1+\sqrt{121})/4 = 3$. The answer is **(a)**.

8. If my daughter is d years old today, and I am e years old today, then the hypothesis gives $d = re$ and $d + 1 = s(e + 1)$. Subtracting these two equations gives $se + s - re = (d + 1) - d = 1$, and therefore $e = (1 - s)/(s - r)$. The answer is **(d)**.

9. Since $\sin 100° = \sin 80°$, $\sin 130° = \sin 50°$, and $\sin 160° = \sin 20°$, while $\sin x$ is an increasing function of x when $0° \leq x \leq 90°$, the answer is **(c)**.

10. The given equation is equivalent to $3^{\mathbf{w}} = \mathbf{cabin}$. There are two integer powers of 3 with 5 digits, namely, $3^9 = 19683$ and $3^{10} = 59049$. Since 59049 does not have distinct digits, we deduce that **cabin** $= 19683$. The answer is **(c)**.

11. Let s denote Bob's speed, and d be the distance from his home to school. Then $d = 20s$. Suppose that x is the required fraction of the way from home to school that Bob covered. Then the hypothesis gives $d(1+x)/s-d(1-x)/s = 7+3 = 10$, which simplifies to $dx = 5s$. Substituting $d = 20s$ into this gives $x = 1/4$. The answer is **(b)**.

12. Suppose that the regular hexagon H has a side of length 1, so that the equilateral triangle T has a side of length 2. Then H can be dissected into 6 equilateral triangles of side 1, by connecting the vertices of H with its center. The triangle T may be decomposed into 4 equilateral triangles of side 1, by cutting it along the 3 line segments which connect the midpoints of its sides to each other. The ratio of the area of T to the area of H is therefore $4/6 = 2/3$. The answer is **(b)**.

13. The total number of possible licence plates for the criminal's car is $c = (3 \cdot 10 \cdot 9) \cdot (26 \cdot 25 \cdot 24)$, while total number of all possible licence plates is $n = 10^3 \cdot 26^3$. We have $c/n = 81/338 \cong 0.239$, so the fraction of licence plates that can be eliminated in the search for the criminal's car is approximately $1 - 0.239 = 0.761$, or 76%. The answer is **(a)**.

14. We must have $\log\sqrt{\log\sqrt{\log x}} \geq 0$, or equivalently $\sqrt{\log\sqrt{\log x}} \geq 1$, or equivalently $\log\sqrt{\log x} \geq 1$, or equivalently $\sqrt{\log x} \geq 10$, or equivalently $\log x \geq 100$, or equivalently $x \geq 10^{100}$. The answer is **(e)**.

15. The center of the circle must lie on the perpendicular bisector of the line segment connecting $(1,0)$ to $(0,1)$, and hence must have coordinates (x,x) for some real number x. By equating the distances of (x,x) to $(1,0)$ and $(-1,-1)$, we see that the radius R of the circle satisfies

$$R^2 = (x-1)^2 + x^2 = (x+1)^2 + (x+1)^2 = 2(x+1)^2.$$

The equation simplifies to $-2x+1 = 4x+2$, which gives $x = -1/6$. Therefore $R^2 = 2 \cdot \left(\frac{5}{6}\right)^2 = \frac{25}{18}$ and the required area is $\pi R^2 = \frac{25\pi}{18}$. The answer is **(a)**.

16. Any harmonic substitution is uniquely determined by the values of A and B, as the sequence A, B, C, D, E must be an arithmetic progression. For A=B we get the ten solutions 00000–99999. For A<B we get the 8 solutions 02468, 13579, and 01234–56789. For A>B we get the reverse 8 solutions 86420, 97531, and 98765–43210. This gives a total of $10 + 8 + 8 = 26$ harmonic substitutions. The answer is **(e)**.

17. The multiples of 17 with 2 digits are 17, 34, 51, 68, and 85, while the multiples of 23 with 2 digits are 23, 46, 69, and 92. Since the last digit of n is 1, the number n must end with the digits $\ldots$923469234692346851. The total number of digits of n, 2015, is a multiple of 5, and, starting from the right, every fifth digit in the above sequence is a 4. The answer is therefore **(c)**.

18. If $|CH|$ is fixed then the angle $\angle ACB$ is largest when $|AC| = |BC|$, so we may assume that ACB is an isosceles triangle. In this case, it is clear that angle ACB is maximal when $|CH|$ is minimal, that is, $|CH| = |AB|/2$. But then ACB is a right triangle with $\angle ACB = 90°$. The answer is **(d)**.

19. The answer to the question is unchanged if we translate both points by the vector $(-100, -200)$, so that one of them coincides with the origin. We may therefore assume that the points are $A(0,0)$ and $B(24, 256)$. A lattice point (x,y) lies on the line through A and B if and only if the slope $\frac{y}{x} = \frac{256}{24} = \frac{32}{3}$. Since 3 and 32 have no common factor larger than 1, we deduce that the lattice points on the line segment AB are the nine points $(0,0), (3,32), (6,64), (9,96), \ldots, (24,256)$. The answer is **(d)**.

20. We have $9991 = 10000 - 9 = 100^2 - 3^2 = (100-3)(100+3) = 97 \cdot 103$. Therefore $p = 103$ and the answer is $1 + 0 + 3 = 4$, namely **(a)**.

21. Suppose that there are n stones with weights $w_1 \leq \cdots \leq w_n$ and let $S := w_1 + \cdots + w_n$. Let m be the largest index such that $w_m + \cdots + w_n > 10$. Then the hypothesis implies that $w_1 + \cdots + w_{m-1} \leq 10$, and hence we have

$$S = (w_1 + \cdots + w_{m-1}) + w_m + (w_{m+1} + \cdots + w_n) \leq 10 + 10 + 10 \leq 30.$$

On the other hand, three stones weighing 10 pounds each give a valid solution. The answer is **(c)**.

22. If $x \geq 0$ then $x + \sqrt{x^2+1} \geq 1$, while $0 < -x + \sqrt{x^2+1} = (x + \sqrt{x^2+1})^{-1} \leq 1$, with equality if and only if $x = 0$. We deduce that x and y are either both zero or have opposite signs, say $x > 0$ and $y < 0$. Then
$$x + \sqrt{x^2+1} = (y + \sqrt{y^2+1})^{-1} = -y + \sqrt{y^2+1}. \tag{1}$$
Since $x + \sqrt{x^2+1}$ is an increasing function of x on the interval $[0, +\infty)$, and x, $-y$ both lie in this interval, we deduce from (1) that $x = -y$, or $x + y = 0$. The answer is **(b)**.

23. Observe that
$$1001 = abc + ab + ac + bc + a + b + c + 1 = (a+1)(b+1)(c+1).$$
Since the prime power decomposition of 1001 is $7 \cdot 11 \cdot 13$, we deduce that $\{a, b, c\} = \{6, 10, 12\}$, and hence that $a + b + c = 28$. The answer is **(b)**.

24. Suppose that $x = k + \epsilon$, where k is an integer and $0 \leq \epsilon < 1$. Then we have $[x] = k$, while $[2x] = 2k$, if $0 \leq \epsilon < 1/2$, and $[2x] = 2k + 1$, otherwise. Similarly $[3x] = 3k$, if $0 \leq \epsilon < 1/3$, $[3x] = 3k + 1$, if $1/3 \leq \epsilon < 2/3$, and $[3x] = 3k + 2$, otherwise. Combining the above cases shows that $n = [x] + [2x] + [3x] = 6k + d$, where $d \in \{0, 1, 2, 3\}$. Therefore the numbers n that are excluded are those integers of the form $6k + 4$ or $6k + 5$ within the interval $[1, 100]$. These are the 33 numbers $4, 5, 10, 11, \ldots, 94, 95, 100$, and so the answer is $100 - 33 = 67$, or **(d)**.

25. The key idea is to work backwards, starting from a $(1, 1, 1)$ triangle and changing one side at a time to arrive at a $(100, 100, 100)$ triangle. The fastest way to do this is as follows. On each move we lengthen the shortest side, making it almost as long as the sum of the other two sides. After the first move we get $(1, 1, 2_{-})$, where 2_{-} denotes a real number less than 2 but very close to 2. After the second move we obtain $(1, 2_{-}, 3_{-})$, and continue in this manner, obtaining triangles whose side lengths are real numbers a little bit smaller than the Fibonacci numbers
$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$$
The procedure ends when the smallest of the three numbers in the triple of side lengths exceeds 100, which occurs after 12 moves.

To prove that the number 12 of moves needed is minimal, let $(a_n \leq b_n \leq c_n)$ be the triangle we obtain after a valid sequence of n moves. It is easy to see that $c_n < c_{n-2} + c_{n-1}$, for any $n \geq 3$. By induction, it follows that $c_n < F_{n+2}$, where F_{n+2} is the $(n+2)$-th Fibonacci number. Since we modify only one side at a time, we have $a_n \leq c_{n-2} < F_n$, and since $F_n < 100$ for $n < 12$, we conclude that after less than 12 moves the length of shortest side a_n will be less than 100. Therefore, there must be at least 12 moves used, and the answer is **(c)**.