THE 38$^{\text{th}}$ ANNUAL (2016) UNIVERSITY OF MARYLAND

HIGH SCHOOL MATHEMATICS COMPETITION

PART I   MULTIPLE CHOICE

For each of the following 25 questions, carefully blacken the appropriate box on the answer sheet with a #2 pencil. Do not fold, bend or write stray marks on either side of the answer sheet. Each correct answer is worth 4 points. **Two points are deducted for each incorrect answer.** *Zero points are given if no box, or more than one box, is marked. Note that wild guessing is apt to lower your score. When the exam is over, give your answer sheet to the proctor. You may keep your copy of the questions.*

NO CALCULATORS

75 MINUTES

1. A hamster is gradually eating a cookie. During the first day it eats 1/3 of the cookie. During the second day it eats an additional 1/4 of the cookie. The third day it eats an additional 1/5 of the cookie, and so on. How many days will it take until the entire cookie is devoured?

   a. 4   b. 5   c. 6   d. 7   e. 8

2. Jackie has a magic board that highlights all 7's (and leaves all other digits intact). One day when she is bored, she writes all integers between 1 and 1000 on her magic board. How many digits are highlighted on her board?

   a. 243   b. 270   c. 271   d. 300   e. 330

3. Newton's doctor says he needs a vacation. He tells him to get a tan lying on a cot in the sun for 7 days. Instead, Newton computes x such that $\dfrac{\tan x}{\cot x} = 7$. When the doctor hears about this, he scolds Newton, who replies that he is aware of his sin. Assuming that x is an acute angle, what is $\sin x$?

   a. $1/7$ b. $\sqrt{1/7}$ c. $\sqrt{2/7}$ d. $\sqrt{7/8}$ e. $\sqrt{1/8}$

4. For a positive real number x we know x percent of x is 0.3. What is $x^4 + x^2 + 1$?

   a. 1.39   b. 13   c. 931   d. 1101   e. 90301

5. The six digit number $x3xx7y$ is divisible by 12. Here x and y represent different digits. What is the value of y?

   a. 0   b. 2   c. 4   d. 6   e. 8

6. Define the exponent sum $ES(n)$ of an integer n to be the sum of the exponents of the primes in the prime factorization of n. For example, $ES(198) = 4$ because $198 = 2^1 \cdot 3^2 \cdot 11^1$ and the sum of the exponents is 4. Compute $ES(2016)$.

   a. 4   b. 5   c. 7   d. 8   e. 9

7. In a zoo, a Tiger lives in a yard that is a triangle with sides of lengths 5, 10, and 10, and with area T. A Crocodile lives in a pond that is a circle of radius 3, and with area C. A Sloth sleeps in a yard that is a square with side length 5, and with area S. All lengths are in meters. Each claims to have the enclosure with the largest area. Which of the following is true?

   a. $T < S < C$ b. $S < T < C$ c. $C < T < S$ d. $T < C < S$ e. $S < C < T$

8. Optimus Prime challenges Sponge Bob Square Pants to find the largest two-digit prime number p such that the sum of the digits of p is a square, and the smallest two-digit prime q such that the sum of the digits of q is a square. What is $p + q$?

   a. 44   b. 92   c. 110   d. 128   e. 176

9. A gym has a collection of n weights, each an odd number of pounds, and no two weighing the same amount. The weights all fit in a box with the capacity to hold 2016 pounds. What is the largest possible value of n under these conditions?

   a. 41   b. 42   c. 43   d. 44   e. 45

10. If $1000^{\log 3} + 1000^{\log 4} + 1000^{\log 5} = 1000^{\log x}$, then x equals (Note: "log" is logarithm in base 10.)

   a. 6   b. 8   c. 10   d. 30   e. 60

11. I have q quarters and d dimes in my pocket, with a total value of $2.00. How many pairs (q, d) of non-negative integers make this possible?

   a. 2   b. 4   c. 5   d. 7   e. 10

12. Four people attend a math contest and they each bring a pencil. To spread around their good luck, they put the pencils in a bag and then each person draws out a pencil. How many possible assignments of the pencils are there such that none of the four contestants obtains their original pencil.

   a. 6   b. 9   c. 11   d. 12   e. 14

13. In triangle ABC, side AB has length 4, side BC has length 5, and side AC has length 6. What is the sine of angle BAC?

   a. 4/5   b. 5/6   c. $5\sqrt{7}/16$ d. $(2 + \sqrt{5})/6$ e. 3/10

14. Consider a polygon P and a point O in the plane. It is known that the rotation with center O and angle θ maps P onto itself (here θ is an angle with $0 < \theta < 2\pi$ in radians). Which of the following *must* be true?

   a. P is a regular polygon   b. P has an even number of sides   c. All sides of P have the same length   d. All angles of P are equal   e. θ is a rational multiple of π radians.

15. Let $f(x) = \sin^4(x) + \cos^4(x)$, where x is in radians. What is the smallest positive real number z such that $f(x + z) = f(x)$ for all real numbers x?

a. $\pi/4$　b. $\pi/2$　c. π　d. $\sqrt[4]{\pi}$　e. $\sqrt[4]{\pi}/2$

16. Let N be the smallest integer such that there is no perfect square n^2 with $N < n^2 < N + 100$. What is the sum of the digits of N?

a. 7　b. 8　c. 11　d. 12　e. 13

17. Let
$$x = \frac{3}{1^2 \cdot 2^2} + \frac{5}{2^2 \cdot 3^2} + \frac{7}{3^2 \cdot 4^2} + \cdots + \frac{4033}{2016^2 \cdot 2017^2}.$$
The decimal expansion of x is $0.999999d54\cdots$. What is the digit d?

a. 0　b. 1　c. 4　d. 5　e. 7

18. Suppose points A and B in a plane are given such that $|AB| = 100$. How many points C in the same plane are there so that A, B, and C form a triangle (with positive area) and all sides of this triangle are integers between 1 and 100, inclusive?

a. 5050　b. 9900　c. 10050　d. 10100　e. 10200

19. In the triangle ABC consider the point M on BC with $|BM| < |CM|$. From M we draw lines parallel to AB and AC. Suppose the area of the resulting parallelogram is 5/18 of the area of ABC. What is the ratio $|BM|/|CM|$?

a. 1/6　b. 1/4　c. 1/5　d. 2/9　e. 1/3

20. For any positive integer n, let $f(n)$ be the number of 1's that appear in the base-2 representation of n. Then, $\sum_{i=1}^{1023} f(n)$ equals

a. 2^{11}　b. $10 \cdot 2^9$　c. $11 \cdot 2^9$　d. $10!$　e. $11!$

21. A positive integer a is called *balanced* if its digits can be divided into two groups with equal sums. The least balanced number a so that $a + 1$ is also balanced satisfies

a. $100 \le a < 200$　b. $200 \le a < 300$　c. $300 \le a < 400$　d. $400 \le a < 500$
e. $500 \le a < 600$

22. Consider an isosceles triangle ABC with $\angle ABC = \angle ACB = 40°$. Let D be a point on line AB such that $|AD| = |BC|$ and B lies between A and D. Then $\angle BCD$ equals

a. 5°　b. 8°　c. 10°　d. 12°　e. 15°

23. Find the number of integers between 1 and 100, inclusive, that can be written as a sum of non-negative integers such that digits 0-9 are all used exactly once (e.g. $90 = 0 + 1 + 52 + 3 + 4 + 6 + 7 + 8 + 9$).

a. 7　b. 11　c. 15　d. 19　e. 23

24. Let $S = \{\frac{1}{256}, \frac{1}{32}, \frac{1}{4}, 2, 16, 128, 1024\}$. How many real numbers can be written as a product of three distinct elements of S?

   a. 35   b. 21   c. 20   d. 15   e. 13

25. A sphere is divided into regions by 9 planes that are passing through its center. What is the largest possible number of regions that are created on its surface?

   a. 2^8 b. 2^9 c. 81   d. 76   e. 74