THE 38$^{\text{th}}$ ANNUAL (2016) UNIVERSITY OF MARYLAND

HIGH SCHOOL MATHEMATICS COMPETITION

PART I SOLUTIONS

1. After four days the hamster eats $1/3 + 1/4 + 1/5 + 1/6 = 19/20$ fraction of the cookie. Since $19/20 + 1/7 > 1$, the hamster needs five days. The answer is **(b)**.

2. The number of 7's in units digit is $10 \times 10 = 100$, since there are ten choices for the tens digit and ten choices for the hundreds digit. The same is true for 7's in the tens digit and the hundreds digits. Therefore, there are $3 \times 100 = 300$ digits that are highlighted. The answer is **(d)**.

3. $\dfrac{\tan x}{\cot x} = 7$. Thus $\tan^2 x = 7$. This implies $\tan x = \sqrt{7}$. Using a right triangle or trigonometric identities we obtain $\sin x = \sqrt{7/8}$. The answer is **(d)**.

4. $x$ percent of $x$ is $x \cdot \dfrac{x}{100} = 0.3$. Thus, $x^2 = 30$. This implies $x^4 + x^2 + 1 = 931$. The answer is **(c)**.

5. Using the divisibility test by 4, the two-digit number $7y$ must be divisible by 4. This yields $y = 2$ or $y = 6$. The sum of digits, $3x + 10 + y$, must be divisible by 3. Therefore $y = 2$. The answer is **(b)**.

6. $2016 = 2^5 \cdot 3^2 \cdot 7$. Therefore, $ES(2016) = 8$. The answer is **(d)**.

7. The altitude to the side of length 5 in the triangle is $\sqrt{10^2 - 2.5^2} = \sqrt{93.75}$. Thus, $T = 5\sqrt{93.75}/2 < 25$. $C = 9\pi > 27$. $S = 25$. Therefore, $T < S < C$. The answer is **(a)**.

8. $p = 97$ and $q = 13$. This gives $p + q = 110$. The answer is **(c)**.

9. To get the smallest total weight we choose the first $n$ odd positive integers. Using the formula for the sum of terms of an arithmetic sequence we get $1+3+\cdots+(2n-1) = \dfrac{1+2n-1}{2} \cdot n = n^2$. We want $n^2 \leq 2016$. Therefore, $n \leq 44$. The answer is **(d)**.

10. We have, $1000^{\log 3} = 10^{3\log 3} = 10^{\log 27} = 27$. Similarly $1000^{\log 4} = 64$ and $1000^{\log 5} = 125$. That gives us $27 + 64 + 125 = x^3$. Therefore $x = 6$. The answer is **(a)**.

11. The assumption gives us $0.25q + 0.1d = 2$. Multiplying by 20 we get $5q + 2d = 40$. Since 5 and 40 are multiples of 5, $d$ must be a multiple of 5. Since 2 and 40 are even, $q$ must be even. This gives us $d = 5m$ and $q = 2n$. Which implies $n + m = 4$. There are 5 pairs: $(n, m) = (0, 4), (1, 3), (2, 2), (3, 1), (4, 0)$. The answer is **(c)**.

12. We are trying to find the number of permutations on $1, 2, 3, 4$ for which no number is fixed. We use Principle of Inclusion and Exclusion. There are $4! = 24$ permutations. From those permutations, there are $3! = 6$ that 1 is fixed. There are $2! = 2$ permutations that fix 1 and 2. There is $1! = 1$ that fix $1, 2$ and 3, and finally there is 1 permutation that fixes all elements. Therefore the answer is $24 - 6 \cdot 4 + 2 \cdot 6 - 1 \cdot 4 + 1 = 9$. The answer is **(b)**.

13. Set $x = \angle BAC$. Using the Law Cosines we get $\cos x = \dfrac{4^2 + 6^2 - 5^2}{2 \cdot 4 \cdot 6} = \dfrac{9}{16}$. Therefore, $\sin x = \sqrt{1 - \dfrac{81}{256}} = 5\sqrt{7}/16$. The answer is **(c)**.

14. The statement is true when $P$ is an equilateral triangle, $O$ is its center and $\theta = \dfrac{2\pi}{3}$. Thus, (b) is false. The statement is also true when $P$ is a parallelogram, $O$ is the intersection of its diagonals and $\theta = \pi$. Thus (a), (c) and (d) are false. The answer is **(e)**.

    **Note**: Lets call this rotation $r$. Consider the powers of this rotation $r, r^2, r^3, \ldots$ (i.e. the composition of this rotation with itself, finitely many times.) Since every rotation is a permutation on vertices, and there are only finitely many permutations of vertices, $r^n = r^m$ for two positive integers $m < n$. This implies $r^{n-m}$ is the identity permutation of vertices. Note that $r^{n-m}$ is the rotation about $O$ with angle $(n-m)\theta$. Since it is the identity permutation, $(n-m)\theta$ must be a multiple of $2\pi$. This implied $\theta$ is a rational multiple of $\pi$.

15. $f(x) = (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x = 1 - \frac{1}{2}\sin^2(2x) = 1 - \frac{1}{4}(1 - \cos(4x))$. The period of this function is $\dfrac{\pi}{2}$. The answer is **(b)**.

16. Suppose $x^2$ is the largest perfect square less than or equal to $N$. By assumption $N+100 \leq (x+1)^2$. Therefore $x^2+100 \leq (x+1)^2$. This implies $100 \leq 2x+1$, which shows $50 \leq x$. Therefore $50^2 \leq N$. We can easily see that $N = 50^2$ works. The answer is **(a)**.

17. Each terms is of form $\dfrac{2n+1}{n^2 \cdot (n+1)^2}$. This is equal to $\dfrac{1}{n^2} - \dfrac{1}{(n+1)^2}$. Thus

    $$x = \left(\frac{1}{1^2} - \frac{1}{2^2}\right) + \left(\frac{1}{2^2} - \frac{1}{3^2}\right) + \cdots + \left(\frac{1}{2016^2} - \frac{1}{2017^2}\right) = 1 - \frac{1}{2017^2}.$$

    The answer is **(e)**.

18. For each pair of positive integers $(m, n)$ for which $m + n > 100$ and $m, n \leq 100$ we get two triangles, one on each side of line $AB$. There are $100 \cdot 100$ pairs $(m, n)$ with $m, n \leq 100$. From those pairs there are $99 + 98 + \cdots + 1$ pairs satisfying $m + n \leq 100$. Thus, the answer is $2 \cdot (10000 - \dfrac{99 \cdot 100}{2}) = 10100$. The answer is **(d)**.

19. Suppose the parallel lines through $M$ intersect $AB$ and $AC$ at $P$ and $Q$ respectively. We know $[ABC] = \dfrac{1}{2}|AB| \cdot |AC| \sin A$, $[APMQ] = |AP| \cdot |AQ| \sin A$. If we set $|BM|/|MC| = x$, then using similar triangles we get $|AP|/|AB| = 1/(1+x)$ and $|AQ|/|AC| = x/(1+x)$. Thus we get

    $$\frac{5}{18} = 2 \cdot \frac{1}{1+x} \cdot \frac{x}{1+x}.$$

    After solving we get $x = 1/5$. The answer is **(c)**.

20. We count the number of digits of 1 that appear in each position. For each position, 1 appears exactly $2^9$ times. This is because base-2 representation of integers between 1 and 1023 have at most 10 digits. Since there are 10 digits, the total number of 1's that appear is $10 \cdot 2^9$. The answer is **(b)**.

21. Note that the sum of digits of any balanced number is even.

The units digit must be 9. Otherwise sum of digits of $a$ and $a+1$ differ by 1, which means they cannot both be even. Now, the smallest possible balanced integer would have to be of form $a = xy9$, where $x + y = 9$. For $a + 1$ to be balanced we must have $x = y + 1$. This implies $y = 4$ and $x = 5$. Thus, the smallest balanced positive integer is 549. The answer is **(e)**.

22. Let $x = \angle BCD$. Use the Law of Sines in triangles $ABC$ and $ADC$. We obtain: $\frac{|AD|}{|AC|} = \frac{\sin(x+40)}{\sin(40-x)}$ and $\frac{|BC|}{|AC|} = \frac{\sin 100}{\sin 40} = \frac{2\sin 50 \cos 50}{\sin 40} = 2\sin 50$. Therefore, $\frac{\sin(x+40)}{\sin(40-x)} = \frac{\sin 50}{\sin 30}$. This suggests $x = 10$. You can see $x = 10$ is the only solution, because $\frac{\sin(x+40)}{\sin(40-x)}$ is a strictly increasing function for $0 < x < 40$. The answer is **(c)**.

23. Clearly every integer with given property is at least $1 + 2 + \cdots + 9 = 45$. Notice that using the divisibility test by 9, each integer satisfying the given property is a multiple of 9. Therefore the possible values are $45, 54, 63, 72, 81$ and 99. It is easy to see all of these values can be obtained. The answer is **(a)**.

24. Notice that $S = \{2^{-8}, 2^{-5}, 2^{-2}, 2, 2^4, 2^7, 2^{10}\}$. Any product of three distinct elements of $S$ is a power of 2, where the exponent is a sum of three distinct elements of $A = \{-8, -5, -2, 1, 4, 7, 10\}$. Notice that elements of $A$ form an arithmetic sequence with common difference 3. Thus, any sum of three distinct elements of $A$ is of form $(3k+1)+(3n+1)+(3m+1) = 3(k+n+m)+3$, where $-3 \leq k, m, n \leq 3$. Note that $k+m+n$ can be any integer between $-3+(-2)+(-1) = -6$ and $3+2+1 = 6$. Therefore, there are 13 possible sums. The answer is **(e)**.

25. The maximum is obtained when no three planes pass through the same diameter of the sphere. Lets denote the surface of the sphere by $S$. One plane divides $S$ into two regions. The second plane intersects the first plane in 2 points on $S$, thus adding two more regions. The third plane intersects each of the previous planes on $S$ at two points which means we will get $2 \times 2$ additional regions. The $n$th plane intersects the previous $n-1$ planes at $2(n-1)$ points, which means it would add $2(n-1)$ regions on $S$. Therefore the total number of regions is $2 + 2 + 2 \times 2 + 2 \times 3 + \cdots + 2 \times 8 = 2 + 2 \times \frac{8 \times 9}{2} = 74$. The answer is **(e)**.