THE 39th ANNUAL (2017) UNIVERSITY OF MARYLAND

HIGH SCHOOL MATHEMATICS COMPETITION

PART I MULTIPLE CHOICE

For each of the following 25 questions, carefully blacken the appropriate box on the answer sheet with a #2 pencil. Do not fold, bend or write stray marks on either side of the answer sheet. Each correct answer is worth 4 points. **Two points are deducted for each incorrect answer.** *Zero points are given if no box, or more than one box, is marked. Note that wild guessing is apt to lower your score. When the exam is over, give your answer sheet to the proctor. You may keep your copy of the questions.*

NO CALCULATORS

75 MINUTES

1. Every day, Alex spends 4 hours on the Internet. Every week, Alex spends 50 hours sleeping. Each year, Alex goes to school for 180 days and spends 8 hours in class each of these days. Let I be the number of hours on the Internet each year. Let S be the number of hours sleeping each year. Let C be the number of hours in class each year. Assume that there are 52 weeks in a year and that there are 365 days in a year. Then,

   a. $C > S > I$ b. $I > S > C$ c. $S > I > C$ d. $I > C > S$ e. $S > C > I$

2. Evaluate $1 - 2 + 3 - 4 + 5 - 6 + \cdots - 2016 + 2017$.

   a. 1009 b. 2017 c. 3025 d. 4033 e. 20160

3. The line $y = 2x + 1$ intersects the circle $x^2 + y^2 = 13$ in two points. One of the intersection points is $(-2, -3)$. The x-coordinate of the other intersection point is

   a. 1 b. 6/5 c. 3/2 d. 2 e. 11/2

4. Pat collects 100 pieces of candy while trick-or-treating. Of these, 75 contain chocolate, 50 contain both chocolate and marshmallow, 10 contain neither chocolate nor marshmallow. How many contain marshmallow but no chocolate?

   a. 15 b. 20 c. 25 d. 30 e. 35

5. Achilles and Zeno compete in a 10-mile race. Zeno walks at a constant speed. For the first 5 miles, Achilles runs with a hare that goes twice as fast as Zeno. For the last 5 miles, Achilles walks with a tortoise that goes half as fast as Zeno. Achilles finishes the race one hour after Zeno. How fast does Zeno walk, in miles per hour?

   a. 2 b. 2.5 c. 4 d. 4.5 e. 5

6. Evaluate $2^{\log_2(3)} + 3^{\log_3(4)} + 4^{\log_4(5)}$.

   a. 3 b. 6 c. 9 d. 10 e. 12

7. Trains from Washington to Baltimore leave Washington every w minutes and always travel at the same speed. A train from Baltimore to Washington also travels at this speed. A passenger on the Baltimore to Washington train notices that they pass a train traveling in the opposite direction every 6 minutes. What is w?

   a. 5 b. 6 c. 10 d. 12 e. 15

8. Ten children eat candy. The oldest child eats 18 more pieces of candy than the average consumption of all 10 children. Each of the other 9 children eats 15 pieces of candy. How many pieces of candy does the oldest child eat?

   a. 20 b. 24 c. 30 d. 33 e. 35

9. The product of all of the positive divisors of 100 (including 100, itself) has the form 10^m. What is the exponent m?

   a. 8 b. 9 c. 10 d. 11 e. 12

10. $\dfrac{\cos 1^\circ + \cos 2^\circ + \cdots + \cos 89^\circ}{\cos 91^\circ + \cos 92^\circ + \cdots + \cos 179^\circ}$ equals

   a. -1 b. 0 c. 1 d. $\frac{\sqrt{2}}{2}$ e. $\sqrt{2}$

11. Consider the sequence $a_1, a_2, a_3, \ldots$, such that $a_1 = 3, a_2 = 7$, and $a_{n+1} = a_n - a_{n-1}$ for all $n \geq 2$. What is the value of a_{2017} ?

   a. -7 b. -4 c. -3 d. 3 e. 4

12. How many integers k satisfy all of the inequalities $|k - 50| < |k - 200| < |k - 10|$?

   a. 17 b. 18 c. 19 d. 20 e. 21

13. For a positive integer n, let $p(n)$ be the largest odd factor of n. For example, $p(60) = 15$ and $p(39) = 39$. Evaluate the sum

$$p(1001) + p(1002) + p(1003) + \cdots + p(1999) + p(2000).$$

   a. 2000 b. 65536 c. 150000 d. 800000 e. 1000000

14. A 2×10 rectangle $ABCD$ is completely covered by ten non-overlapping 1×2 rectangles. In how many ways is it possible to do this?

   a. 55 b. 73 c. 75 d. 80 e. 89

15. Suppose X and Y are sets of real numbers, each of size 100, such that the average values of the numbers in the sets X and Y are -10 and 10, respectively; and the average values of the numbers in the sets $X \cup Y$ and $X \cap Y$ are 1 and -4, respectively. What is the size of $X \cap Y$?

   a. 10 b. 30 c. 40 d. 70 e. 90

16. Suppose that points A, B, C are spaced in clockwise order on a circle of radius one, for which $|AB| = |BC| = 1.2$. What is $|AC|$?

    a. $\sqrt{3}$ b. 1.81 c. 1.86 d. 1.92 e. 1.96

17. From three vertices A, B and C of a square $ABCD$ we draw three parallel lines ℓ_1, ℓ_2 and ℓ_3, respectively, such that ℓ_2 lies between ℓ_1 and ℓ_3. Suppose the distance between ℓ_1 and ℓ_2 is 5 and the distance between ℓ_2 and ℓ_3 is 7. What is the area of the square $ABCD$?

    a. 64 b. 70 c. 74 d. 81 e. 112

18. If $x^2 + y^2 + z^2 + w^2 = x^4 + y^4 + z^4 + w^4 = 1$ for some real numbers x, y, z, and w, then what is the largest possible value of $x + y + z + w$?

    a. 1 b. $\sqrt{2}$ c. $2\sqrt{2}$ d. $\sqrt[4]{2}$ e. $\sqrt[4]{8}$

19. Consider three pairwise tangent circles, each of radius 1. What is the area of the equilateral triangle formed by three lines, each of which is tangent to two of the circles and does not intersect the third?

    a. $4\sqrt{3} + 6$ b. $64/5$ c. 12 d. $3\sqrt{3} + 9$ e. $9\sqrt{2}$

20. Suppose for three real numbers a, b, c we know that $a + b = 2c - 2$ and $ab = 2c^2 + c - 3$. What is the sum of the maximum and minimum possible values of c?

    a. -3 b. -1 c. 0 d. 1 e. 3

21. In an isosceles triangle ABC, we know $|AB| = |AC|$. Point D on side AC is selected so that BD is the angle bisector of B. Suppose $|BC| = |AD| + |BD|$. What is the angle A, in degrees?

    a. 100 b. 108 c. 110 d. 115 e. 120

22. Let $f(x) = x^3 + 4x - 8$. The solution set to $-552 < f(f(x)) < f(4x)$ is given by $a < x < b$. What is $b - a$?

    a. 1 b. 2 c. 3 d. 4 e. 5

23. Let $f(x) = ax^2 + bx + c$, where a, b, c are three real numbers. Suppose for all x, $0 \leq x \leq 1$, we have $|f(x)| \leq 1$. What is the maximum possible value of $2a + b$?

    a. 6 b. 7 c. 8 d. 9 e. 10

24. Suppose that $f(x) = \dfrac{4^x}{4^x + 2}$. Find the value of $f(\frac{1}{14}) + f(\frac{2}{14}) + \cdots + f(\frac{13}{14})$.

    a. 7 b. 8.5 c. 6.5 d. 4.5 e. 5.5

25. The equation $x^6 - 15x^4 + 20x^3 - 30x^2 + 8 = 0$ has four real roots $r_1 < r_2 < r_3 < r_4$. The value of $r_1 r_2 + r_3 r_4$ is closest to

    a. 4.4 b. 4.3 c. 4.2 d. 4.1 e. 4