THE 39$^{\text{th}}$ ANNUAL (2017) UNIVERSITY OF MARYLAND

HIGH SCHOOL MATHEMATICS COMPETITION

PART I SOLUTIONS

1. By assumption $I = 365 \times 4 = 1460$, $S = 50 \times 52 = 2600$ and $C = 180 \times 8 = 1440$. The answer is **(c)**.

2. If we pair up the numbers we obtain:

$$(1-2)+(3-4)+\cdots+(2015-2016)+2017 = (-1)\times 1008 + 2017 = 1009$$

The answer is **(a)**.

3. Substituting $y = 2x+1$ into the equation of the circle, we get $x^2 + (2x+1)^2 = 13$. Simplifying we get $5x^2 + 4x - 12 = 0$. Factoring we get $(5x-6)(x+2) = 0$. Therefore $x = -2, 6/5$. The answer is **(b)**.

Note: Since you know $x = -2$ is a solution you can find the other solution using the fact that the product of the two solution is $-12/5$. This is one of the so-called Vieta's Formulas.

4. We know there are 10 candies that contain neither marshmallow nor chocolate. We also know that there are 75 that contain chocolate. Therefore, there are $100 - 75 - 10 = 15$ candies that contain marshmallow but no chocolate. The answer is **(a)**.

5. Suppose s is the speed of Zeno. The time that it would take Zeno to finish the race is $10/s$. The speed of Achilles in the first 5 miles of the race is $2s$ and in the second 5 miles it is $\frac{s}{2}$. Therefore, the time that it takes Achilles to finish the race is $\frac{5}{2s} + \frac{5}{s/2}$.

By assumption, we have $\frac{10}{s} + 1 = \frac{5}{2s} + \frac{5}{s/2} = \frac{5}{2s} + \frac{10}{s}$. Thus $\frac{5}{2s} = 1$, which implies $s = 2.5$. The answer is **(b)**.

6. By properties of log, $a^{\log_a b} = b$. Therefore $2^{\log_2(3)} + 3^{\log_3(4)} + 4^{\log_4(5)} = 3+4+5 = 12$. The answer is **(e)**.

7. Let s be the speed of each train. In 6 minutes a train from Baltimore to Washington travels half the distance between two trains that travel from Washington to Baltimore. Therefore $w = 6 \times 2 = 12$. The answer is **(d)**.

8. Suppose x is the average consumption of all 10 children. Then the oldest child consumes $18+x$. By assumption we have $\frac{(18+x)+15\times 9}{10} = x$. Therefore, $18 + x + 15 \times 9 = 10x$, which implies $x = 17$. The oldest child eats $18 + 17 = 35$ pieces of candy. The answer is **(e)**.

9. Positive divisors of 100 are 1, 2, 4, 5, 10, 20, 25, 50 and 100. The product is 10^9. The answer is **(b)**.

10. Note that $\cos(180-x) = -\cos x$. Using this identity we see that $\cos 1^\circ = -\cos 179^\circ, \ldots, \cos 89^\circ = -\cos 91^\circ$. The answer is **(a)**.

11. Writing several terms of the sequence we see the terms repeat after the 6th term:

$$3, 7, 4, -3, -7, -4, 3, 7, \ldots$$

Since the remainder of 2017 when divided by 6 is 1, we have $a_{2017} = a_1 = 3$. The answer is **(d)**.

12. **Geometric solution.** The inequality $|k - 50| < |k - 200|$ implies k is closer to 50 than to 200. This means k is less than $\dfrac{50 + 200}{2} = 125$. Similarly, the inequality $|k - 200| < |k - 10|$ implies k is closer to 200 than to 10. This means k is more than $\dfrac{10 + 200}{2} = 105$. Thus $105 < k < 125$. The number of such integers is $125 - 105 - 1 = 19$.

Algebraic solution. Squaring we get $k^2 - 100k + 2500 < k^2 - 400k + 40000 < k^2 - 20k + 100$. Therefore $300k < 37500$ and $380k > 39900$. Therefore $105 < k < 125$.

The answer is **(c)**.

13. Let $0 < k < 2000$ be odd. We claim that precisely one integer of form $2^r k$ appears in the list $1001, 1002, \ldots, 2000$. If r is the largest integer satisfying $2^r k \leq 1000$, then $1000 < 2^{r+1}k \leq 2000$ and that $2000 < 2^{r+2}k$. This proves our claim. This means every positive odd integer less than 2000 appears once in the list $p(1001), p(1002), \ldots, p(2000)$. Therefore the answer is $1 + 3 + \cdots + 1999 = \dfrac{1 + 1999}{2} \times 1000 = 1000000$. The answer is **(e)**.

14. Let a_n be the number of ways a $2 \times n$ rectangle can be tiled with 1×2 tiles. Clearly $a_1 = 1$ and $a_2 = 2$. To get a tiling for a $2 \times n$ rectangle we can either tile a $2 \times (n-1)$ rectangle and add a 1×2 rectangle or tile a $2 \times (n-2)$ rectangle and add two 1×2 rectangles. Thus, $a_n = a_{n-1} + a_{n-2}$. Therefore the sequence a_n is $1, 2, 3, 5, 8, 13, 21, 34, 55, 89$ The answer is **(e)**.

15. Suppose $X \cap Y$ has n elements. Since X and Y have 100 elements, $X \cup Y$ has $200 - n$ elements. Let $X = \{x_1, \ldots, x_{100}\}$ and $Y = \{y_1, \ldots, y_{100}\}$. By assumption $x_1 + \cdots + x_{100} + y_1 + \cdots + y_{100} = 0$. This sum has all elements of $X \cup Y$, with the ones in $X \cap Y$ each appearing twice. By assumption, the sum of elements of $X \cap Y$ is $-4 \cdot n$ and the sum of elements of $X \cup Y$ is $1 \cdot (200 - n)$. Therefore $-4n + 200 - n = 0$. Therefore $n = 40$. The answer is **(c)**.

16. By assumption ABC is an isosceles triangle. Let $x = \angle BAC = \angle BCA$. Note that $\angle ABC = 180 - 2x$. By extended law of sines we have $\dfrac{1.2}{\sin x} = \dfrac{|AC|}{\sin(180 - 2x)} = 2$. This implies $\sin x = 0.6$, thus $\cos x = 0.8$. Therefore, $\sin(180 - 2x) = \sin(2x) = 2 \sin x \cos x = 2 \times 0.6 \times 0.8 = 0.96$. This implies $\dfrac{|AC|}{0.96} = 2$, therefore $|AC| = 1.92$. The answer is **(d)**.

17. Let x be the side length of the square. From B, we draw a line perpendicular to ℓ_1 and ℓ_3. This creates two right triangles with AB and BC as their hypotenuses. The two right triangles are congruent. Therefore $x^2 = 5^2 + 7^2 = 74$. The answer is **(c)**.

18. By assumption, $x^2 \leq 1$, which implies $|x| \leq 1$. This implies $x^4 \leq x^2$; similar for y, z and w. Therefore $x^4 + y^4 + z^4 + w^4 \leq x^2 + y^2 + z^2 + w^2$. Since the equality holds, we have $x^2 = x^4$, which implies $x^2 = 0, 1$; similar for y^2, z^2 and w^2. Since $x^2 + y^2 + z^2 + w^2 = 1$, one of x^2, y^2, z^2, w^2 is one and the others are zero. Thus the maximum of $x + y + z + w$ is 1. The answer is **(a)**.

19. Suppose ABC is the triangle formed by the three tangent lines. Let P and Q be points of tangency of AB with the circles, so that P is between A and Q. Let O_1 and O_2 be the centers of circles closer to A and B, respectively. We know $|PQ| = |O_1O_2| = 2$. Since $\angle O_1AP = 30°$, $AP = \sqrt{3}$. Thus, $AB = 2+2\sqrt{3}$. This implies the area of ABC is $\frac{\sqrt{3}}{4}(2+2\sqrt{3})^2 = \sqrt{3}(4+2\sqrt{3}) = 4\sqrt{3}+6$. The answer is **(a)**.

20. Not that in general, given real numbers S and P, there are real numbers a and b for which $a+b=S$ and $ab=P$, iff the equation $x^2 - Sx + P = 0$ has two real roots, which is true iff the discriminant is non-negative. In other words $S^2 - 4P \geq 0$. Therefore

$$(2c-2)^2 - 4(2c^2+c-3) \geq 0 \Rightarrow 4c^2-8c+4-8c^2-4c+12 = -4c^2-12c+16 \geq 0 \Rightarrow c^2+3c-4 \leq 0$$

Therefore, $-4 \leq c \leq 1$. The answer is **(a)**.

21. **First Solution.** We use trigonometry. Let $x = \angle DBA = \angle DBC$. Thus, $\angle ABC = \angle ACB = 2x$, which implies $\angle BDC = 180 - 3x$ and $\angle BAC = 180 - 4x$. Using the Law of Sines in triangles ABD and BDC, we obtain $\frac{|AD|}{|BD|} = \frac{\sin x}{\sin(180-4x)}$ and $\frac{|BC|}{|BD|} = \frac{\sin(180-3x)}{\sin(2x)}$. By assumption $\frac{|AD|}{|BD|} + 1 = \frac{|BC|}{|BD|}$. Using the identities that we found above we conclude,

$$\frac{\sin x}{\sin(180-4x)} + 1 = \frac{\sin(180-3x)}{\sin(2x)} \Rightarrow \frac{\sin x + \sin(4x)}{\sin(4x)} = \frac{\sin(3x)}{\sin(2x)} \Rightarrow \frac{\sin x + \sin(4x)}{2\sin(2x)\cos(2x)} = \frac{\sin(3x)}{\sin(2x)}.$$

Multiplying by $\sin(2x)$ and using the formula $2\sin a \cos b = \sin(a+b) + \sin(a-b)$ we get the following:

$$\sin x + \sin(4x) = 2\cos(2x)\sin(3x) = \sin(5x) + \sin(x) \Rightarrow \sin(4x) = \sin(5x).$$

Therefore $4x + 5x = 180 \Rightarrow x = 20$. This implies $\angle BAC = 180 - 4 \times 20 = 100$.

Second Solution. This is a less computational but more clever solution. Select point E on side BC such that $|CE| = |AD|$. By assumption, $|BD| = |BE|$. By Angel-Bisector Theorem, $\frac{|AB|}{|BC|} = \frac{|AD|}{|DC|} = \frac{|CE|}{|CD|}$. Therefore the triangle ABC and ECD are similar by SAS, which implies $\angle EDC = \angle DCE$. Similar to the first solution let $x = \angle ABD$. We see that $\angle CDE = 2x$, since DEC is isosceles. $\angle ADE = 90 - x/2$, since BDE is isosceles, and $\angle ADB = 3x$. Therefore $3x+90-x/2+2x = 180$. Hence, $x = 20$. Similar to the first solution we obtain $\angle BAC = 100$.

The answer is **(a)**.

22. Note that if $x < y$, then $x^3 < y^3$, therefore f is increasing. Note also that $f(-8) = -552$, which implies $-8 < f(x) < 4x$. Therefore $0 < x^3 + 4x$ and $x^3 < 8$. Hence, $0 < x < 2$. The answer is **(b)**.

23. We see that $f(0) = c$, $f(1) = a+b+c$ and $f(1/2) = a/4+b/2+c$. Solving this system we obtain that $2a+b = f(0)+3f(1)-4f(1/2)$. Thus, $2a+b \leq 8$. Equality holds for $f(x) = 8x^2-8x+1$. The answer is **(c)**.

24. For every x we have, $f(1-x) = \frac{4^{1-x}}{2+4^{1-x}} = \frac{4}{2\cdot 4^x + 4} = \frac{2}{4^x+2}$. This shows that $f(x)+f(1-x) = 1$. Therefore,

$$f(1/14) + f(13/14) = f(2/14) + f(12/14) = \cdots = f(6/14) + f(8/14) = 2f(7/14) = 1.$$

The answer is **(c)**.

25. **First solution.** Let$f(x)$ be the given polynomial. Note that if x is a solution, so is $\frac{2}{x}$. This can be verified by substituting $\frac{2}{x}$ for x. We also see that $f(0) > 0$ and $f(-1) < 0$, therefore the graph of f crosses the x-axis at least once between $x = 0$ and $x = -1$. Let r be such a root of f between -1 and 0. We know $2/r < -2$ and that $2/r$ is another root of $f(x)$. Similarly f has a root between 0 and 1 and one larger than 2. By the discussion above $r_1r_2 = r_3r_4 = 2$. Therefore $r_1r_2 + r_3r_4 = 4$.

Second Solution. Similar to above, solutions come in pairs $(r, 2/r)$. This suggests substituting $S = x + \frac{2}{x}$. This implies $S^3 = x^3 + \frac{8}{x^3} + 6x + \frac{12}{x} = x^3 + \frac{8}{x^3} + 6S$. Dividing the equation by x^3, we obtain

$$x^3 + \frac{8}{x^3} - 15(x + \frac{2}{x}) + 20 = 0 \Rightarrow S^3 - 6S - 15S + 20 = 0 \Rightarrow (S-1)(S-4)(S+5) = 0$$

Therefore $S = 1, 4, -5$. Solving each we get four real solutions $x = 2 \pm \sqrt{2}, \frac{-5 \pm \sqrt{17}}{2}$.

The answer is **(e)**.