THE 40^{th} ANNUAL (2018) UNIVERSITY OF MARYLAND

HIGH SCHOOL MATHEMATICS COMPETITION

PART I MULTIPLE CHOICE

For each of the following 25 questions, carefully blacken the appropriate box on the answer sheet with a #2 pencil. Do not fold, bend or write stray marks on either side of the answer sheet. Each correct answer is worth 4 points. **Two points are deducted for each incorrect answer.** *Zero points are given if no box, or more than one box, is marked. Note that wild guessing is apt to lower your score. When the exam is over, give your answer sheet to the proctor. You may keep your copy of the questions.*

NO CALCULATORS

75 MINUTES

1. The sum of four consecutive integers is 26. What is their product?

   a. 120 b. 360 c. 840 d. 1680 e. 3024

2. George decides to build a tower 555 feet high. He has blocks of height 30 feet and blocks of height 21 feet. What is the smallest number of blocks that he can use?

   a. 20 b. 22 c. 24 d. 26 e. 28

3. Evaluate $\sin((60\cos(120°))°)$.

   a. -1/2 b. 1/2 c. $\sqrt{3}/2$ d. $-\sqrt{3}/2$ e. 0

4. Gauss tells Riemann to add some logs to the fire, so Riemann computes

   $$4+\log(3)+\log(2/3)+\log(5)$$

   (logs are logarithms to base 10.)

   What answer does he get?

   a. 3 b. 4 c. 5 d. 6 e. 7

5. Emma and Ethan are two students in a class of 24 students. The average of all 24 final exam scores in their class is 75. The average of all other 22 scores except Emma and Ethan's is 73. What is the average score of Emma and Ethan?

   a. 74 b. 84 c. 87 d. 95 e. 97

6. What is the area of the triangle formed by the lines $x=2$, $y=3$ and $2y+3x=18$ in the xy-plane?

   a. 3/2 b. 2 c. 5/2 d. 3 e. 6

7. Every day at 12:00 and at 6:00, the hour hand and the minute hand of a clock lie on a straight line. In each day, how many times between 12:01 pm and 11:59 pm do the minute hand and the hour hand of a clock lie on a straight line?

   a. 11 b. 12 c. 22 d. 23 e. 24

8. Suppose that a and b are two real numbers. If the lines with equations $ax + 2y = 7$ and $10x + by = 5$ are the same line, then a equals

   a. 10 b. 14 c. 9 d. 15 e. 5

9. Alex walks from school to home at a constant rate of 4 feet per second. One day, he leaves school at 3:00 pm. After walking for one minute, he rests for 1 second. Then he continues walking for another minute, and then rests for 2 seconds. Then he walks for another minute and rests for 3 seconds. This pattern of walking for a minute and resting a second longer each time continues until he gets home. Alex arrives at home precisely at 3:18 pm. What is the distance between Alex's school and his home in feet?

   a. 3720 b. 3840 c. 3960 d. 4080 e. 4280

10. The number 100000 is factored as a product of two positive integers m and n. Suppose that neither m nor n has the digit 0 in its base 10 expansion. What is $m + n$?

    a. 1100 b. 2011 c. 2718 d. 3142 e. 3157

11. Let $S = \{1, 2, \ldots, 10\}$. A nonempty subset X of S is said to be *balanced* if it has the following property:

    "If x is an element of X, then $(11 - x)$ is also an element of X."

    The number of nonempty balanced subsets of S is

    a. 1 b. 15 c. 16 d. 31 e. 32

12. Which of the triangles whose side lengths are listed below has the largest area?

    a. 15, 20, 23 b. 15, 20, 24 c. 15, 20, 25 d. 15, 20, 26 e. 15, 20, 29

13. Suppose that a convex quadrilateral $ABCD$ is such that the line segments AC and BD are perpendicular. Let O be the point of intersection between AC and BD, and suppose that $[ABO] = 12, [BCO] = 14$, and $[CDO] = 21$, where $[XYZ]$ denotes the area of $\triangle XYZ$. What is $[DAO]$?

    a. 16 b. 17 c. 18 d. 19 e. 20

14. Suppose that x and y are distinct real numbers such that $3x^2 + 3y^2 = 10xy$. The absolute value of the quotient $(x + y)/(x - y)$ is equal to

    a. 2 b. 3 c. 5 d. 6 e. 7

15. A line ℓ passes through the vertex A of square $ABCD$, and has no other points in common with the square. It is known that the side AB has length 1 and forms an angle of 30 degrees with ℓ. The shortest distance from the vertex C to the line ℓ is equal to

    a. $2(\sqrt{3}+1)/3$ b. $(\sqrt{3}+1)/2$ c. $\sqrt{3}$ d. $3(\sqrt{3}+1)/4$ e. $\sqrt{3}+1$

16. How many pairs of nonnegative integers (m, n) satisfy the inequality $10m + 99n \leq 990$?

    a. 250 b. 347 c. 399 d. 450 e. 551

17. Suppose $n_0, n_1, n_2, \ldots$ is a sequence of integers satisfying *all* of the following:

    - $0 \leq n_k \leq 123$ for every $k \geq 0$.
    - For every $k \geq 0$, n_{k+1} is the remainder when $2n_k + 1$ is divided by 124.
    - $n_1 \neq n_0$.

    What is the least possible value of $\ell > 0$ for which $n_\ell = n_0$ for some such sequence?

    a. 2 b. 3 c. 4 d. 5 e. 123

18. For every positive integer n, let

    $$a_n = 1 + \sqrt{\frac{1}{n} - \sqrt{\frac{1}{n+1} - \sqrt{\frac{1}{n} - \frac{1}{n+1}}}}.$$

    Then the product $a_1 a_2 \cdots a_{99}$ equals

    a. 9/1000 b. 2/95 c. 1/55 d. 1/91 e. 1/40

19. A unit cube is a cube whose volume is 1. What is the volume of an octahedron whose vertices are centers of six faces of a unit cube?

    a. $\sqrt{2}/3$ b. $1/6$ c. $\sqrt{2}/2$ d. $\sqrt{6}/8$ e. $\sqrt{3}/6$

20. In triangle ABC points D and E on side $\overline{AC}$ are chosen so that $|AD| : |DE| : |EC| = 1 : 2 : 3$. Points F and G on side $\overline{AB}$ are so that $DF \parallel EG \parallel CB$. Suppose the numerical value of the perimeter of ADF is equal to the numerical value of the area of $EGBC$. What is the inradius of the triangle ABC (i.e. the radius of the circle inscribed in triangle ABC)?

    a. 2/9 b. 1/3 c. 4/9 d. 1/2 e. 2/3

21. For how many integers n between 1 and 2018, inclusive, does there exist a positive real number x for which $x\lfloor x \rfloor = n$? ($\lfloor x \rfloor$ is the largest integer not exceeding x.)

    a. 972 b. 978 c. 984 d. 988 e. 990

22. Let $A = (2, 3)$ and $B = (4, -1)$ be two points in the xy-plane. Let S be the set of all points C in the xy-plane for which ABC is a triangle whose angles are $30°, 60°$ and $90°$ (in some order). What is the sum of x-coordinates of all points in S?

    a. 36 b. 32 c. 30 d. 28 e. 24

23. For every positive integer n let $f(n)$ be the number of zeros at the end of $n!$ when $n!$ is written in base 10. For example $f(4) = 0$, $f(5) = 1$, and $f(11) = 2$. Which of the following is the number of integers $a \geq 0$ for which there is a positive integer $n \leq 2018$ with $f(n) = a$?

a. 403 b. 404 c. 405 d. 503 e. 504

24. Find the sum of all positive real numbers x, for which $\sqrt[3]{2+x} + \sqrt[3]{2-x}$ is an integer.

a. $\sqrt{2}$ b. $\sqrt[3]{4}$ c. $\sqrt{5}$ d. $\dfrac{3\sqrt{15}+10}{3\sqrt{3}}$ e. $\dfrac{\sqrt{5}(\sqrt{3}+\sqrt{2}+1)}{\sqrt{3}}$

25. How many pairs of (positive or negative) integers (x, y) satisfy the following equation?

$$3x^2 + 16xy + 21y^2 = 2,000,000$$

a. 112 b. 98 c. 96 d. 84 e. 56