THE 40$^{\text{th}}$ ANNUAL (2018) UNIVERSITY OF MARYLAND HIGH SCHOOL MATHEMATICS COMPETITION

PART I SOLUTIONS

1. If the smallest of the four consecutive integers is x, then we must have $x + (x + 1) + (x + 2) + (x + 3) = 26$. Thus, $4x + 6 = 26$, which implies $x = 5$. So, the product of the four integers is $5 \times 6 \times 7 \times 8 = 1680$. The answer is **d**.

2. Suppose George uses x blocks of height 30 feet and y blocks of height 21 feet. We must have $30x + 21y = 555$. Since 30 and 555 are divisible by 5, the integer y must also be divisible by 5. If we set $y = 0$, we get $x = \frac{111}{6}$ which is not an integer. Thus, the smallest value of y is 5. This yields $x = 15$ and $y = 5$. The answer is **a**.

3. $\sin((60\cos(120°))°) = \sin((60 \times (-1/2))°) = \sin(-30°) = -1/2$. The answer is **a**.

4. By properties of log we have

$$4 + \log(3) + \log(2/3) + \log(5) = 4 + \log(3 \times 2/3 \times 5) = 4 + \log(10) = 4 + 1 = 5.$$

   The answer is **c**.

5. The total score in this class is $24 \times 75 = 1800$. The total score of everybody except for Ethan and Emma is $22 \times 73 = 1606$. Therefore the total score of Ethan and Emma is $1800 - 1606 = 194$. Therefore the average score of Ethan and Emma is 97. The answer is **e**.

6. The lines $x = 2$ and $2y + 3x = 18$ intersect at $(2, 6)$. The lines $y = 3$ and $2y + 3x = 18$ intersect at $(4, 3)$. The lines $x = 2$ and $y = 3$ intersect at $(2, 3)$. Therefore the two legs of this right triangle are of length 3 and 2. This means the area of the triangle is $3 \times 2/2 = 3$. The answer is **d**.

7. Normally in each hour there are two instances that the hour hand and the minute hand of the clock lie on a straight line. The two exceptions are between 12:01 pm and 1 pm, between 11 pm and 11:59 pm, in which there is only one instance that the minute hand and the hour hand lie on a straight line. However the instance 6 pm is counted twice, once between 5 pm and 6 pm and once between 6 pm and 7 pm. Thus the answer is $12 \times 2 - 3 = 21$.

   Note: This answer choice is not listed. We have given full credit to all students for this problem.

8. The x-intercepts of the two lines are $7/a$ and $5/10 = 1/2$. Therefore, we must have $7/a = 1/2$, which implies $a = 14$. The answer is **b**.

9. Right after the k-th time Alex walks one minute he will have spent $(60+1)+(60+2)+(60+3)+\cdots+(60+(k-1))+60 = 60k + \frac{k(k-1)}{2}$ seconds walking and resting combined. We are looking for the smallest k for which this amount is at least $18 \times 60 = 1080$. We can see that $k = 16$ gives us precisely 1080 seconds. Therefore, Alex will have spent 16 minutes walking. Since his speed is 4 ft/sec, the distance between his school and home is $16 \times 60 \times 4 = 3840$ ft. The answer is **b**.

10. We have $100000 = 2^5 \times 5^5$. Neither m nor n can have both 2 and 5 as factors. Therefore, m and n must be 2^5 and 5^5. Thus $m + n = 32 + 3125 = 3157$. The answer is **e**.

11. Note that from each pair of numbers $(1, 10), (2, 9), \ldots, (5, 6)$ either both numbers are in X or neither is in X. Therefore, including the empty set, there are 2^5 possible such sets. Removing the empty set we get 31 balanced subsets. The answer is **d**.

12. Note that the area of a triangle ABC is evaluated by $\frac{1}{2}|AB| \cdot |AC| \sin A$. Thus, if the sides AB and AC are of lengths 15 and 20, the area is maximized when $A = 90°$. In other words the area is maximized when the triangle is a right triangle. The answer is **c**.

13. Note that $2[ABO] = |AO| \cdot |BO|$, $2[CDO] = |CO| \cdot |DO|$, $2[BCO] = |BO| \cdot |CO|$, and $2[ADO] = |AO| \cdot |DO|$. This implies $[ABO] \cdot [CDO] = [BCO] \cdot [ADO]$. Therefore $12 \times 21 = 14 \times [ADO]$. The answer is **c**.

14. $3x^2 - 10xy + 3y^2 = 0$. By the quadratic formula we get $x = \frac{10y \pm \sqrt{100y^2 - 36y^2}}{6} = \frac{10y \pm 8y}{6} = 3y, \frac{y}{3}$. Therefore $\frac{x}{y} = 3, 1/3$. We have $(x + y)/(x - y) = (3y + y)/(3y - y) = 2$ or $x = (y/3 + y)/(y/3 - y) = -2$. The answer is **a**.

15. Let H be the foot of the perpendicular from C to ℓ. We know $\angle HAC = 75°$. Therefore $|CH| = |AC| \sin(75)$. We know $|AC| = \sqrt{2}$. We also have

$$\sin(75) = \sin(45 + 30) = \sin(45)\cos(30) + \cos(45)\sin(30) = \frac{\sqrt{6} + \sqrt{2}}{4}$$

Therefore, $|CH| = (\sqrt{3} + 1)/2$. The answer is **b**.

16. We first find all points (m, n) where m, n are nonnegative integers that lie on the line $10m + 99n = 990$. Note that $10m + 99n = 990 \Longrightarrow 99 \mid m$ and $10 \mid n$ (since 99 and 10 are relatively prime). If m and n were both positive, we would have $10m + 99n \geq 2 \cdot 990$, which is a contradiction. Thus the only nonnegative solutions to $10m + 99n = 990$ are those in which one of m and n are zero — specifically, $(99, 0)$ and $(0, 10)$.

    Let B denote the box region in the plane defined by

    $$B = \{(x, y) \mid 0 \leq x \leq 99, 0 \leq y \leq 10\},$$

    which contains $11 \cdot 100 = 1100$ integer points. The line $10x + 99y = 990$ connects the two vertices $(99, 0)$ and $(0, 10)$ and splits B into two triangles, B_1 and B_2. By symmetry, there are an equal number of integer points in B_1 and B_2. Moreover, as noted above, there are only two integer points that lie in both B_1 and B_2. Thus the desired answer is

    $$(1100 - 2)/2 + 2 \quad = \quad 551.$$

    The correct answer is **e**.

17. We will work mod 124. Using the properties of this sequence we obtain: $n_1 \equiv 2n_0 + 1 \mod 124$. Thus, $n_2 \equiv 2(2n_0 + 1) + 1 = 2^2 n_0 + 2 + 1$, which implies $n_3 = 2^3 n_0 + 2^2 + 2 + 1$. We can see that $n_\ell = 2^\ell n_0 + 2^\ell - 1$. Note that $n_\ell \equiv n_0 \mod 124$ iff $124 \mid (2^\ell - 1)(n_0 + 1)$. Since $124 = 4 \cdot 31$, we have $4 \mid (n_0 + 1)$. Therefore $n_0 + 1 \equiv 4$ or $0 \mod 124$. The latter is impossible,

because otherwise $n_0 \equiv -1 \mod 124$, which implies $n_1 \equiv 2(-1)+1 = -1 \mod 124$, which is a contradiction. Therefore $n_0 + 1 \equiv 4 \mod 124$, which means $31 \mid 2^\ell - 1$, which implies $\ell \geq 5$. For $n_0 = 3$, we get the sequence $3, 7, 15, 31, 63, 3$. The answer is **d**.

18.

$$\begin{aligned}
\prod_{i=1}^{99}\left[1+\left(\frac{1}{\sqrt{i}}-\frac{1}{\sqrt{i+1}}\right)-\sqrt{\frac{1}{i}-\frac{1}{i+1}}\right] &= \prod_{i=1}^{99}\left[1+\frac{1}{\sqrt{i}}-\frac{1}{\sqrt{i+1}}-\sqrt{\frac{(i+1)-i}{i(i+1)}}\right] \\
&= \prod_{i=1}^{99}\left[1+\frac{1}{\sqrt{i}}-\frac{1}{\sqrt{i+1}}-\frac{1}{\sqrt{i(i+1)}}\right] \\
&= \prod_{i=1}^{99}\left(\left[1+\frac{1}{\sqrt{i}}\right]\left[1-\frac{1}{\sqrt{i+1}}\right]\right) \\
&= \left[1+\frac{1}{\sqrt{1}}\right]\left(\prod_{i=2}^{99}\left[1+\frac{1}{\sqrt{i}}\right]\left[1-\frac{1}{\sqrt{i}}\right]\right)\left[1-\frac{1}{\sqrt{100}}\right] \\
&= [2]\left(\prod_{i=2}^{99}\left[1-\frac{1}{i}\right]\right)\left[\frac{9}{10}\right] \\
&= [2]\left(\prod_{i=2}^{99}\left[\frac{i-1}{i}\right]\right)\left[\frac{9}{10}\right] \\
&= [2]\left(\frac{1}{99}\right)\left[\frac{9}{10}\right] \\
&= \frac{1}{55}
\end{aligned}$$

The answer is **c**.

19. This octahedron can be divided into two pyramids. Since the distance between centers of opposite faces of cube is 1, the height of each pyramid is $1/2$. The base of each pyramid is a square whose side length is the distance between centers of two adjacent faces. Let F_1 and F_2 be two adjacent faces of this cube. From the centers C_1 and C_2 of F_1 and F_2 drop perpendiculars to the side that is shared by F_1 and F_2. Let H be the foot of this perpendicular. We have $|C_1H| = |C_2H| = 1/2$. Thus by the Pythagorean Theorem $|C_1C_2|^2 = 1/2$, . Therefore, the volume of each pyramid is $\frac{1}{3}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{12}$, which implies the volume of the octahedron is $1/6$. The answer is **b**.

20. Let s and A be the semi-perimeter and area of triangle ABC, respectively. We know the inradius of ABC is A/s. Since triangle ADF and ACB are similar with the ratio of similarity $1/6$, the perimeter of ADF is $2s/6 = s/3$. Since AEG and ACB are similar with the ratio of similarity $3/6$, the area of ADF is $A/4$. Therefore, by assumption we have $s/3 = 3A/4$, which simplifies to $4s = 9A$. Therefore, $A/s = 4/9$. The answer is **c**.

21. Let $\lfloor x \rfloor = k$. Then we have $k^2 \leq x\lfloor x \rfloor < (k+1)k$. Therefore $k^2 \leq n < k^2 + k$. Thus, for a given k, there are $k^2 + k - k^2 = k$ possible values of n. Note that $44 \times 45 = 1930 < 2018$ but $45^2 = 2025 > 2018$. Thus the answer is $\sum_{k=1}^{44} k = \frac{44 \times 45}{2} = 990$. The answer is **e**.

22. There are four such triangles with $A = 90°$, two on each side of line AB. Two of these four are reflections of the other two about the line AB. Thus, the sum of the x-coordinates of these four points C is four times the x-coordinate of A, which is $4 \times 2 = 8$.

    Using the same logic for when $B = 90°$ we obtain $4 \times 4 = 16$.

    There are also four more such triangles with $C = 90°$, two on each side of line AB. The two on one side of AB are obtained by reflecting the other two about the midpoint of $\overline{AB}$. The midpoint of $\overline{AB}$ is $(3, 1)$. Thus the sum of the x-coordinates of these points is $4 \times 3 = 12$.

    The total of all x-coordinates is $8 + 16 + 12 = 36$, thus the answer is **a**.

23. Note that $f(1) = f(2) = f(3) = f(4)$, however the value of f changes at 5 and remains the same until $n = 9$, i.e. $f(5) = \cdots = f(9)$. Then it changes and stays the same until $n = 14$. Thus the value of f changes precisely $\lfloor 2018/5 \rfloor = 403$ times. Since it starts from 0, we obtain $1 + 403 = 404$. The answer is **b**.

24. Let $n = \sqrt[3]{2+x} + \sqrt[3]{2-x}$. Note that $n > \sqrt[3]{x} + \sqrt[3]{-x} = 0$. Cube both sides to obtain $n^3 = 2+x+2-x+3\sqrt[3]{4-x^2}(\sqrt[3]{2+x}+\sqrt[3]{2-x}) = 4+3n\sqrt[3]{4-x^2}$. This shows $n^3 < 4+3n\sqrt[3]{4}$. Thus, $n^2 < 4/n + 3\sqrt[3]{4} \leq 4 + 5 = 9$, which is only possible for $n = 1, 2$.

    If $n = 1$, then $1 = 4 + 3\sqrt[3]{4-x^2}$. This implies $4 - x^2 = -1$, which shows $x = \sqrt{5}$.

    If $n = 2$, then $8 = 4+6\sqrt[3]{4-x^2}$. This implies $4-x^2 = 8/27$, which shows $x = \sqrt{100/27} = \dfrac{10}{3\sqrt{3}}$.

    The sum of the two possible values of x is $\sqrt{5} + \dfrac{10}{3\sqrt{3}}$. The answer is **d**.

25. Factoring the quadratic we obtain $(x+3y)(3x+7y) = 2^7 \times 5^6$. Note that $(x+3y)+(3x+7y) = 4x+10y$ is even. Thus $x+3y$ and $3x+7y$ are both even. If $x+3y = 2a$ and $3x+7y = 2b$, then $x = 3b - 7a$ and $y = 3a - b$. Therefore we need to find the number of pairs of integers (a, b) for which $(2a)(2b) = 2^7 \times 5^6$, or equivalently $ab = 2^5 \times 5^6$. The number of positive divisors of $2^5 \times 5^6$ is $6 \times 7 = 42$. The number of all divisors is 84. The answer is **d**.