THE 41^{st} ANNUAL (2019) UNIVERSITY OF MARYLAND

HIGH SCHOOL MATHEMATICS COMPETITION

PART I SOLUTIONS

1. Writing all of the answer choices with base 2, we have $4^{12} = 16^6 = 2^{24}$, $32^5 = 2^{25}$, and $8^9 = 2^{27}$. The answer is **e**.

2. Sam gives Carlos x^2. Carlos gives Fiona x^6, and Fiona evaluates $\sqrt[4]{x^6} = \sqrt{x^3} = 1000$. Thus, $x^3 = 1000^2 = 10^6$, which implies $x = 10^2 = 100$. The answer is **c**.

3. The cockroach crosses the x-axis each time $y = 0$. Thus, we need to solve $x^5 - x = 0$. This implies $x(x^4 - 1) = 0$. Thus, $x = 0$, or $x^4 = 1$. Therefore, $x = 0, \pm 1$. The answer is **c**.

4. $6! = 2 \cdot 3 \cdot 2^2 \cdot 5 \cdot 2 \cdot 3 = 2^4 \cdot 3^2 \cdot 5$. Using the properties of log, we obtain

$$\log(6!) = 4\log(2) + 2\log(3) + \log(5).$$

   Thus, $a + b + c = 4 + 2 + 1 = 7$. The answer is **d**.

5. Multiplying the first equation by 3 and adding it to the second equation we obtain $5y = 5$, and thus $y = 1$, and therefore, $x = 1$. This means $(1, 1)$ must be on the last line, which implies $5 + k = 7$, and thus $k = 2$. The answer is **a**.

6. Since N is not a prime, it must have at least two prime factors. The smallest primes more than 100 are 101 and 103. Since N is not a perfect square N cannot be 101^2. Thus, $N = 101 \cdot 103 = 10403$ is the smallest such integer. The answer is **b**.

7. The angle sum of an n-gon is $180(n-2)$. Each angle of a regular n-gon is $180(n-2)/n$ degrees, which is equal to $180 - \dfrac{360}{n}$. So, 360 must be a multiple of n. Therefore, the largest value of n is 360. The answer is **b**.

8. By assumption, the number of peaches that Sammy picks must be both a multiple of 5 and also a multiple of 3. Thus, Sammy must pick $15k$ peaches, where k is a positive integer. By assumption Sally and Sunny must pick $3k$ and $5k$ peaches, respectively. Therefore, the total number of peaches that they all picked together must be $15k + 3k + 5k = 23k$, which is a multiple of 23. The largest multiple of 23 less than 200 is $23 \cdot 8 = 184$. The answer is **a**.

9. Using the sum of the terms of an arithmetic sequence we obtain $17 + 18 + \cdots + 84 = \dfrac{17 + 84}{2} \cdot 68 = 101 \cdot 34$. Similarly $1 + 4 + \cdots + 100 = \dfrac{1 + 100}{2} \cdot 34 = 101 \cdot 17$. Thus the desired ratio is 2. The answer is **c**.

10. Using the Pythagorean Theorem, we have $d^2 + 4000^2 = (4000 + 1/2)^2 = 4000^2 + 4000 + 1/4$. Thus, $d = \sqrt{4000.25}$, which is between 60 and 70. The answer is **d**.

11. Suppose the rate of drinking water by the elephant is r liters per minute and the volume of the tank is V. Using the formula time $= \dfrac{\text{volume}}{\text{rate}}$ and the given information we obtain the following

equations:

$$\begin{cases} \dfrac{V}{r-4} - \dfrac{V}{r-2} = 40 \\[2ex] \dfrac{V}{r-2} - \dfrac{V}{r-1} = 10 \end{cases}$$

This implies $V(r-2) - V(r-4) = 40(r-2)(r-4)$, and $V(r-1) - V(r-2) = 10(r-1)(r-2)$. Therefore, $2V = 40(r-2)(r-4)$ and $V = 10(r-1)(r-2)$. Dividing the two equations we get $2 = 4(r-4)/(r-1)$, which implies $r-1 = 2(r-4)$, and hence $r = 7$. Therefore, $V = 20(7-2)(7-4) = 300$. The answer is **e**.

12. If the units digit is 0, then there are 9 possibilities for the hundreds digit and 8 possibilities for the tens digit. Thus we get $9 \times 8 = 72$ possible integers. If the units digit is $2, 4, 6$, or 8, then there are 8 possibilities for the hundreds digit and also 8 possibilities for the tens digit. This gives us $4 \times 8 \times 8 = 256$ possibilities. Adding the two we get $72 + 256 = 328$. The answer is **c**.

13. Removing the cubes with the icing we get a rectangular prism of size $(n-1) \times (n-2) \times (n-2)$. Since $(n-2)^2$ is a perfect square, $n-1$ must be a perfect square. The answer is **e**.

14. We know $\sin(x) = \cos(\pi/2 - x)$. Given $\cos(\pi/2 - x) = \cos(3x)$, we obtain $\pi/2 - x = 2\pi k \pm 3x$, which implies $x = \pi k - \pi/4$ or $x = -\pi k/2 + \pi/8$. Given $0 \leq x \leq 2\pi$, we obtain $x = 3\pi/4, 7\pi/4, \pi/8, 5\pi/8, 9\pi/8, 13\pi/8$. The answer is **b**.

15. After s seconds, the hour hand moves $s \cdot 30/3600 = s/120$ degrees. The minute hand moves $s \cdot 30/300 = s/10$ degrees. The second hand moves $6s$ degrees. Using the assumption, $6s - s/120 = 90 + s/10 - 6s$. Solving this we get $s = 10800/1427$. The answer is **b**.

16. Note that the two right triangles ABE and ADF are congruent, by (HL). Therefore, $\angle EAB = 15°$. Thus, $AE = \sec(15°)$. Since AEF is equilateral, each altitude of AEF is $\sqrt{3}AE/2$. Thus,

$$[AEF] = \frac{\sqrt{3}}{4}AE^2 = \frac{\sqrt{3}}{4\cos^2(15°)} = \frac{\sqrt{3}}{2(1+\cos(30°))} = \frac{\sqrt{3}}{2+\sqrt{3}} = 2\sqrt{3} - 3.$$

The answer is **a**.

17. We must have

$$100a + 10b + c = 196c + 14b + a \Rightarrow 99a = 4b + 195c.$$

Since 99 and 195 are both divisible by 3, b must be a multiple of 3 as well. Setting $b = 3d$, we obtain $33a = 4d + 65c$, which implies $4d - c = 33(a - 2c)$. Therefore, $4d - c$ must be divisible by 33. Since, $-10 < 4d - c < 16$, we must have $4d = c$. This implies $33a = 4d + 260d = 264d$, which implies $a = 8d$. Since $0 < a < 10$, we must have $d = 1$, and thus $b = 3$. The answer is **b**.

Note: The desired number is 834.

18. Unfolding the south and north walls of the room we can find the shortest distance from the initial to the terminal point of the path by considering a segment connecting them. After unfolding, the ant must move 120 units in the y direction and 50 units in the x direction. Thus the shortest distance is $\sqrt{50^2 + 120^2} = 130$. The answer is **e**.

19. The two cells in the first column can be colored in 6 different ways. For each coloring of a column, there are precisely three valid coloring of the next column. Therefore the answer is $6 \cdot 3^{99} = 2 \cdot 3^{100}$. The answer is **c**.

20. Let O_1, O_2, and O be centers of C_1, C_2, and ω, respectively. Let C be the point of intersection of ω with ℓ. Drop a perpendicular from O_1 to OC and call the foot of this perpendicular H. Using the Pythagorean Theorem in triangle O_1HO, we obtain $|AC|^2 + (r-1)^2 = (r+1)^2$, which implies $|AC| = 2\sqrt{r}$. Similarly $|BC| = 8\sqrt{r}$. Now, if C lies between A and B, we have $|AB| = |AC| + |BC|$. Otherwise $|AB| = |BC| - |AC|$. Therefore, $20 = 10\sqrt{r}$ or $20 = 6\sqrt{r}$, which implies $r = 4$ or $r = 100/9$. Thus, the sum of the two possible radii is $136/9$. The answer is **b**.

21. Define a new sequence b_n by $b_n = a_n + 0.5$ for every $n \geq 1$. By assumption $b_n = a_{n-1}^2 + a_{n-1} - 1.75 = (a_{n-1} + 0.5)^2 - 2 = b_{n-1}^2 - 2$. We know $b_1 = 2 + 2^{-1}$, $b_2 = (2 + 2^{-1})^2 - 2 = 2^2 + 2^{-2}$, $b_3 = (2^2 + 2^{-2})^2 - 2 = 2^4 + 2^{-4}$. We can see that $b_n = 2^{2^{n-1}} + 2^{-2^{n-1}}$. (This can be proved by induction.) Therefore, $a_{10} = 0.5 + 2^{2^9} + 2^{-2^9} \approx 0.5 + 2^{2^9}$. The answer is **c**.

22. Note that $1.2 : 1.6 : 2 = 3 : 4 : 5$, and thus ABC is a right triangle. Assume $\angle BAC = 90°$, $\angle ABC = \alpha$, $\angle ACB = \beta$, $|AC| = 1.2$, and $|AB| = 1.6$. Let M be the midpoint of BC. Let ω_1, ω_2, and ω_3 be three unit circles centered at A, B and C, respectively. Note that ω_2 and ω_3 are tangent and thus, there is only one point , i.e. M, that is within 1 unit of B and C. We need to find the area of the region inside both ω_1 and ω_2. Let N be the other point of intersection of ω_1 and ω_2. The area of sector BMN is $\dfrac{2\alpha}{2\pi} \cdot \pi = \alpha$. The area of triangle BMN is $0.6 \times 0.8 = 0.48$. Therefore, the answer is $2\alpha - 0.96 + 2\beta - 0.96 = 2(\alpha + \beta) - 1.92 = \pi - 1.92$. The answer is **e**.

23. There are several possible ways that the cricket could be at or above 5 after 5 moves.

    Case I. All moves are to the right. The probability of that happening is $\left(\dfrac{2}{3}\right)^5 = \dfrac{32}{3^5}$.

    Case II. There are four moves of +2 units and one move of -1 unit. The probability of that happening is $\left(\dfrac{1}{3}\right)^5 \cdot 5 = \dfrac{5}{3^5}$.

    Case III. There are three moves of +2 units, one move of +1 unit and one move of -1 unit. The probability of that happening is $\left(\dfrac{1}{3}\right)^5 \cdot 5 \cdot 4 = \dfrac{20}{3^5}$.

    Case IV. There are two moves of +2 units, two move of +1 unit and one move of -1 unit. The probability of that happening is $\left(\dfrac{1}{3}\right)^5 \cdot 5 \cdot \binom{4}{2} = \dfrac{30}{3^5}$.

    The answer is $\dfrac{32 + 5 + 20 + 30}{3^5} = \dfrac{87}{3^5} = \dfrac{29}{81}$. The answer is **d**.

24. Let H be the circumcenter of triangle DPA. Note that EF, and GH are perpendicular bisectors of AP and CP, respectively. Therefore, $EF \parallel GH$ and that the distance between EF and GH is half of $|AC|$. Similarly $FG \parallel HE$ and the distance between FG and HE is half of $|BD|$. Since $EFGH$ is a parallelogram, we have $[EFGH] = 60$. We have $[EFGH] = \dfrac{|AC|}{2}|EF| \geq |AC| \cdot |BD|/4$, which implies $|AC| \cdot |BD| \leq 240$. We have

    $$[ABCD] = [ABD] + [CBD] \leq |AP| \cdot |BD|/2 + |CP| \cdot |BD|/2 = |AC| \cdot |BD|/2 \leq 120.$$

    The equality occurs when $ABCD$ is a square. The answer is **d**.

25. Let $f(x,y) = x^3 - y^3 + x^2y - xy^2 - 2xy - 7x^2 + 5y^2 + 11x + y - 5$. Note that $f(0,y) = -y^3 + 5y^2 + y - 5$. We can see that $f(0,1) = f(0,-1) = f(0,5) = 0$. Therefore,

$$f(0,y) = -(y-1)(y+1)(y-5) = (y-1)(-y-1)(y-5).$$

Similarly we can see that

$$f(x,0) = (x-1)^2(x-5).$$

Matching the two, we may guess that $f(x,y)$ has factors of $x+y-1, x-y-1$, and $x+y-5$. Indeed we can see that

$$x^3 - y^3 + x^2y - xy^2 - 2xy - 7x^2 + 5y^2 + 11x + y - 5 = (x+y-1)(x-y-1)(x+y-5).$$

Thus, $f(x,y) = 0$, implies $x+y=1$, or $x-y=1$, or $x+y=5$. The first one has two solutions, the second has 100 solutions and the last one has 6 solutions. The first and second have exactly one common solution, the second and last also have one common solution. Thus, the answer is $2 + 100 + 6 - 1 - 1 = 106$. The answer is **e**.