26th ANNUAL UNIVERSITY OF MARYLAND
HIGH SCHOOL MATHEMATICS COMPETITION
PART II
December 1, 2004, 1:00–3:00

NO CALCULATORS
2 hours

1. (30 points) Archimedes, Euclid, Fermat, and Gauss had a math competition. Archimedes said, "I did not finish 1st or 4th." Euclid said, "I did not finish 4th." Fermat said, "I finished 1st." Gauss said, "I finished 4th." There were no ties in the competition, and exactly three of the mathematicians told the truth. Who finished first and who finished last? Justify your answers.

2. (30 points) Find the area of the set in the xy-plane defined by $x^2 - 2|x| + y^2 \leq 0$. Justify your answer.

3. (30 points) There is a collection of 2004 circular discs (not necessarily of the same radius) in the plane. The total area covered by the discs is 1 square meter. Show that there is a subcollection S of discs such that the discs in S are non-overlapping and the total area of the discs in S is at least 1/9 square meter.

4. (30 points) Let S be the set of all 2004-digit integers (in base 10) all of whose digits lie in the set $\{1, 2, 3, 4\}$. (For example, $12341234 \cdots 1234$ is in S.) Let n_0 be the number of $s \in S$ such that s is a multiple of 3, let n_1 be the number of $s \in S$ such that s is one more than a multiple of 3, and let n_2 be the number of $s \in S$ such that s is two more than a multiple of 3. Determine which of n_0, n_1, n_2 is largest and which is smallest (and if there are any equalities). Justify your answers.

5. There are 6 members on the Math Competition Committee. The problems are kept in a safe. There are ℓ locks on the safe and there are k keys, several for each lock. The safe does not open unless all of the locks are unlocked, and each key works on exactly one lock. The keys should be distributed to the 6 members of the committee so that each group of 4 members has enough keys to open all of the ℓ locks. However, no group of 3 members should be able to open all of the ℓ locks.
(a) (15 points) Show that this is possible with $\ell = 20$ locks and $k = 60$ keys. That is, it is possible to use 20 locks and to choose and distribute 60 keys in such a way that every group of 4 can open the safe, but no group of 3 can open the safe.
(b) (15 points) Show that we always must have $\ell \geq 20$ and $k \geq 60$.