28^{th} ANNUAL UNIVERSITY OF MARYLAND HIGH SCHOOL MATHEMATICS COMPETITION PART II SOLUTIONS

1. In this problem, a half deck of cards consists of 26 cards, each labeled with an integer from 1 to 13. There are two cards labeled 1, two labeled 2, two labeled 3, etc. A certain math class has 13 students. Each day, the teacher thoroughly shuffles a half deck of cards and deals out two cards to each student. Each student then adds the two numbers on the cards received, and the resulting 13 sums are multiplied together to form a product P. If P is an even number, the class must do math homework that evening. Show that the class always must do math homework.

   Solution: One way: Since there are 14 odd numbers in the half deck, at least one of the 13 students must get two odd numbers, hence an even sum. Therefore the product of all the sums is even. *Another way:* Since each integer appears twice, the sum of all the numbers on the cards is even. If each student had an odd sum, the sum of all the cards would be the sum of 13 odd numbers, hence odd. Therefore, some student must have an even sum, so the product P is even.

2. Twenty-six people attended a math party: Archimedes, Bernoulli, Cauchy, ..., Yau, and Zeno. During the party, Archimedes shook hands with one person, Bernoulli shook hands with two people, Cauchy shook hands with three people, and similarly up through Yau, who shook hands with 25 people. How many people did Zeno shake hands with? Justify that your answer is correct and that it is the only correct answer.

   Solution: Yau shook hands with 25 people, so he shook with everyone A through Z (except Y). Since A shook with exactly one person, it was with Y. Since X shook with 24, and he did not shake with A, he shook with all the remaining 24 people B through Z (except X). Therefore, B shook with exactly X and Y. We now have C through W to work with. W shook with 23, therefore with all of C through V and with X, Y, and Z. This accounts for all of C's shakes. Consider the kth person, counting back from Y. In this way, we keep finding that this person shook with everyone from the kth person, counting from A, through Z, and that all possible handshakes have been counted for everyone A trhough the kth person (counting from A).Eventually, we arrive at the fact that N (this is $k = 12$) shook hands with everyone L through Z, and that M (this is $k = 13$) shook hands with everyone N through Z. At this point, all handshakes have been accounted for, and Z has shaken with everyone M through Y. This is 13 shakes. The answer is **13**.

3. Prove that there are no integers $m, n \geq 1$ such that

$$\sqrt{m+\sqrt{m+\sqrt{m+...+\sqrt{m}}}} = n,$$

where there are 2006 square root signs.

Solution: Suppose such m, n exist. Squaring and subtracting m shows that that the left side, with only 2005 square roots, is an integer. Repeating this procedure 2003 more times, we find that $\sqrt{m+\sqrt{m}}$ is an integer. Call it a. Then $m+\sqrt{m}=a^2$, so $\sqrt{m}$ is an integer, call it b. Therefore, $b^2+b=a^2$. But

$$b^2 < b^2+b < (b+1)^2,$$

so b^2+b cannot be a perfect square. This contradiction shows that m, n cannot exist.

4. Let c be a circle inscribed in a triangle ABC. Let ℓ be the line tangent to c and parallel to AC (with $\ell \neq AC$). Let P and Q be the intersections of ℓ with AB and BC, respectively. As ABC runs through all triangles of perimeter 1, what is the longest that the line segment PQ can be? Justify your answer.

Solution: Let s be the length of BP and let t be the length of BQ. Let D be the intersection of AB and the circle, let E be the intersection of BC and the circle, and let F be the intersection of AC and the circle. Then $AD = AF$ and $CE = CF$, so $AC = AD + CE$. Also, $PQ = PD + QE$ (external tangents have the same length). By assumption, the perimeter is 1, so

$$1 = s + t + PD + QE + AD + CE + AC = s + t + PQ + 2AC.$$

Since PQ is parallel to AC, triangles BPQ and BAC are similar. Therefore, the ratio of their perimeters is the ratio of their corresponding sides:

$$\frac{s+t+PQ}{1} = \frac{PQ}{AC}.$$

This yields

$$1 - 2AC = \frac{PQ}{AC}, \quad \text{so} \quad \mathrm{PQ} = \mathrm{AC} - 2\mathrm{AC}^2.$$

The parabola $y = x - 2x^2$ has it maximum at $x = 1/4$. The value of y is $1/8$. Therefore, the largest possible value for PQ is $1/8$. (This value occurs: If we take $AC = 1/4$ and draw a triangle of perimeter 1, then the above shows that $PQ = 1/8$.)

5. Each positive integer is assigned one of three colors. Show that there exist distinct positive integers x, y such that x and y have the same color and $|x - y|$ is a perfect square.

   Solution: Suppose x and y do not exist. Let $a \geq 10$. Then a, $a - 9$, and $a + 16$ must be different colors since they pairwise differ by squares. Since $a + 7$ differs from $a - 9$ and $a + 16$ by squares, it must be the same color as a. We have shown that a and $a + 7$ must always have the same color, for each $a \geq 10$. Therefore, a, $a + 7$, $a + 14$, $\ldots$, $a + 49$ all have the same color. But a and $a + 49$ differ by a square. Contradiction. Therefore, x and y exist.