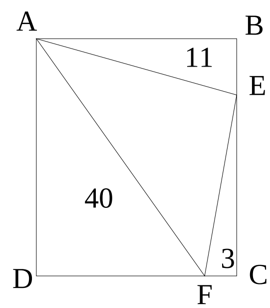
31st ANNUAL UNIVERSITY OF MARYLAND
HIGH SCHOOL MATHEMATICS COMPETITION
PART II
December 2, 2009, 1:00–3:00

NO CALCULATORS
2 hours

1. (a) Show that for every set of three integers, we can find two of them whose average is also an integer.
(b) Show that for every set of 5 integers, there is a subset of three of them whose average is an integer.

2. Let $f(x) = x^2 + ax + b$ and $g(x) = x^2 + cx + d$ be two different quadratic polynomials such that $f(7) + f(11) = g(7) + g(11)$.
(a) Show that $f(9) = g(9)$.
(b) Show that $x = 9$ is the only value of x where $f(x) = g(x)$.

3. Consider a rectangle $ABCD$ and points E and F on the sides BC and CD, respectively, such that the areas of the triangles ABE, ECF, and ADF are 11, 3, and 40, respectively. Compute the area of rectangle $ABCD$.



4. How many ways are there to put markers on a 8×8 checkerboard, with at most one marker per square, such that each of the 8 rows and each of the 8 columns contain an odd number of markers?

5. A robot places a red hat or a blue hat on each person in a room. Each person can see the colors of the hats of everyone in the room except for his own. Each person tries to guess the color of his hat. No communication is allowed between people and each person guesses at the same time (so no timing information can be used, for example). The only information a person has is the color of each other person's hat.

Before receiving the hats, the people are allowed to get together and decide on their strategies. One way to think of this is that each of the n people makes a list of all the possible combinations he could see (there are 2^{n-1} such combinations). Next to each combination, he writes what his guess will be for the color of his own hat. When the hats are placed, he looks for the combination on his list and makes the guess that is listed there.

(a) Prove that if there are exactly two people in the room, then there is a strategy that guarantees that always at least one person gets the right answer for his hat color.

(b) Prove that if you have a group of 2008 people, then there is a strategy that guarantees that always at least 1004 people will make a correct guess.

(c) Prove that if there are 2009 people, then there is no strategy that guarantees that always at least 1005 people will make a correct guess.