

32nd ANNUAL UNIVERSITY OF MARYLAND
HIGH SCHOOL MATHEMATICS COMPETITION

PART II

December 1, 2010, 1:00–3:00

NO CALCULATORS

2 hours

1. We say that six positive integers form a magic triangle if they are arranged in a triangular array as in the figure below in such a way that each number in the top two rows is equal to the sum of its two neighbors in the row directly below it. The triangle shown is magic because $4 = 1 + 3$, $5 = 3 + 2$, and $9 = 4 + 5$.

$$\begin{array}{ccc} & & 9 \\ & 4 & 5 \\ 1 & 3 & 2 \end{array}$$

- (a) Find a magic triangle such that the numbers at the three corners are 10, 20, and 2010, with 2010 at the top.
- (b) Find a magic triangle such that the numbers at the three corners are 20, 201, and 2010, with 2010 at the top, or prove that no such triangle exists.
2. (a) The equalities $\frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1$ and $\frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{42} = 1$ express 1 as a sum of the reciprocals of three (respectively four) distinct positive integers. Find five positive integers $a < b < c < d < e$ such that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} = 1.$$

- (b) Prove that for any integer $m \geq 3$, there exist m positive integers $d_1 < d_2 < \cdots < d_m$ such that

$$\frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_m} = 1.$$

3. Suppose that $P(x) = a_n x^n + \cdots + a_1 x + a_0$ is a polynomial of degree n with real coefficients. Say that the real number b is a *balance point* of P if for every pair of real numbers a and c such that b is the average of a and c , we have that $P(b)$ is the average of $P(a)$ and $P(c)$. Assume that P has two distinct balance points. Prove that n is at most 1, i.e., that P is a linear function.
4. A roller coaster at an amusement park has a train consisting of 30 cars, each seating two people next to each other. 60 math students want to take as many rides as they can, but are told that there are two rules that cannot be broken. First, all 60 students must ride each time, and second, no two students are ever allowed to sit next to each other more than once. What is the maximal number of roller coaster rides that these students can take? Justify your answer.
5. Let $ABCD$ be a convex quadrilateral such that the lengths of all four sides and the two diagonals of $ABCD$ are rational numbers. If the two diagonals AC and BD intersect at a point M , prove that the length of AM is also a rational number.