33rd ANNUAL UNIVERSITY OF MARYLAND
HIGH SCHOOL MATHEMATICS COMPETITION
PART II
November 30, 2011, 1:00–3:00

NO CALCULATORS
2 hours

1. You are given three buckets with a capacity to hold 8, 5, and 3 quarts of water, respectively. Initially, the first bucket is filled with 8 quarts of water, while the remaining two buckets are empty. There are no markings on the buckets, so you are only allowed to empty a bucket into another one or to fill a bucket to its capacity using the water from one of the other buckets.

   (a) Describe a procedure by which we can obtain exactly 6 quarts of water in the first bucket.

   (b) Describe a procedure by which we can obtain exactly 4 quarts of water in the first bucket.

2. A point in the plane is called a lattice point if its coordinates are both integers. A triangle whose vertices are all lattice points is called a lattice triangle. In each case below, give explicitly the coordinates of the vertices of a lattice triangle T that satisfies the stated properties.

   (a) The area of T is $1/2$ and two sides of T have length greater than 2011.

   (b) The area of T is $1/2$ and the three sides of T each have length greater than 2011.

3. Alice and Bob play several rounds of a game. In the nth round, where $n = 1, 2, 3, \ldots$, the loser pays the winner 2^{n-1} dollars (there are no ties). After 40 rounds, Alice has a profit of \$2011 (and Bob has lost \$2011). How many rounds of the game did Alice win, and which rounds were they? Justify your answer.

4. Each student in a school is assigned a 15-digit ID number consisting of a string of 3's and 7's. Whenever x and y are two distinct ID numbers, then x and y differ in at least three entries. Show that the number of students in the school is less than or equal to 2048.

5. A triangle ABC has the following property: there is a point P in the plane of ABC such that the triangles PAB, PBC and PCA all have the same perimeter and the same area. Prove that:

   (a) If P is not inside the triangle ABC, then ABC is a right-angled triangle.

   (b) If P is inside the triangle ABC, then ABC is an equilateral triangle.