THE 33^{rd} ANNUAL (2011) UNIVERSITY OF MARYLAND

HIGH SCHOOL MATHEMATICS COMPETITION

PART II SOLUTIONS

1. (a) Let the 3-tuple (a, b, c) denote the state with a, b, and c quarts of water in the first, second, and third bucket, respectively. Then the initial state is represented by $(8, 0, 0)$, and a procedure which achieves 6 quarts in the first bucket is $(8, 0, 0) \to (3, 5, 0) \to (3, 2, 3) \to (6, 2, 0)$.

   (b) From the position $(6, 2, 0)$ obtained in part (a), we can proceed as follows:

$$(6, 2, 0) \to (6, 0, 2) \to (1, 5, 2) \to (1, 4, 3) \to (4, 4, 0).$$

2. First Solution:

   (a) Let $n = 2011$. The triangle with vertices at $A(0, 0)$, $B(n, 1)$, and $C(n + 1, 1)$ has two sides AB and AC of length greater than n. The area of ABC is clearly $1/2$, as its side BC has length 1 and the height subtending to BC is also equal to 1.

   (b) Consider the point $D(2n + 1, 2)$. Then

$$\vec{AB} + \vec{AC} = (n, 1) + (n + 1, 1) = (2n + 1, 2) = \vec{AD},$$

   and therefore the quadrilateral $ABCD$ is a parallelogram. It is easy to check that the three sides of triangle ABD each have length greater than n. Furthermore, since CD is parallel to AB, the area of ABD equals the area of ABC, and so is again $1/2$.

   Second Solution to (b): We use the formula for the area E of a triangle T with vertices at (x_1, y_1), (x_2, y_2), and (x_3, y_3), namely,

$$E = \frac{1}{2} \left| \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \right| . \tag{1}$$

   We may as well take $(x_3, y_3) = (0, 0)$, in which case (1) gives $E = |x_1 y_2 - x_2 y_1|/2$. From this it is easy to determine the two remaining vertices of T so that the required conditions hold, for example one can take $(x_1, y_1) = (1, n)$ and $(x_2, y_2) = (n, n^2 + 1)$, where $n = 2011$.

3. Let A denote the set of rounds i which Alice won, and B the set of rounds j which Bob won. Since Alice has an end profit of \$2011, we have

$$\sum_{i \in A} 2^{i-1} - \sum_{j \in B} 2^{j-1} = 2011. \tag{2}$$

   On the other hand, since there are exactly 40 rounds, we have

$$\sum_{i \in A} 2^{i-1} + \sum_{j \in B} 2^{j-1} = 1 + 2 + 2^2 + \cdots + 2^{39} = 2^{40} - 1. \tag{3}$$

   Adding equations (2), (3) and dividing the result by 2 gives

$$\sum_{i \in A} 2^{i-1} = \frac{1}{2}(2^{40} + 2010) = 2^{39} + 1005 = 2^{39} + 2^9 + 2^8 + 2^7 + 2^6 + 2^5 + 2^3 + 2^2 + 2^0$$

   where the last equality is equivalent to the binary expansion 1111101101 of 1005. We conclude that Alice wins in nine rounds, namely rounds $1, 3, 4, 6, 7, 8, 9, 10$, and 40.

4. Let us call a 15-digit string of 3's and 7's a 'word'. If x and y are any two words, let $d(x,y)$ denote the number of entries in which x and y differ. Say that there are n ID numbers $x_1,\ldots,x_n$ which are assigned to the n students in the school. For each i with $1 \le i \le n$, there are exactly 16 words y which differ from x_i in at most one entry, i.e., such that $d(x_i,y) \le 1$. Let S_i denote the set of these 16 words. For any $i \ne j$, there is no word y such that both $d(x_i,y) \le 1$ and $d(x_j,y) \le 1$, since then x_i and x_j would differ in at most two entries, contrary to our assumption. It follows that $S_i \cap S_j = \emptyset$ for any $i \ne j$, in other words that the sets S_i are mutually disjoint. Therefore the total number of words contained within these sets (which is $16n$) is less than or equal to the number of all possible words, which is 2^{15}. Since $16n \le 2^{15}$, we conclude that $n \le 2^{11} = 2048$.

5. (a) The lines AB, BC, and AC divide the plane into seven regions, only one of which has finite area (the one which bounds the interior of triangle ABC). If P is outside of ABC, then it must lie in one of the three unbounded regions which share a side with ABC (otherwise one of the triangles PAB, PBC, and PCA would contain the other two). Suppose that P lies outside of ABC but in between the two rays $\vec{AB}$ and $\vec{AC}$. Then triangles PBA and PBC share a side PB and have the same area. We deduce that points A and C are equidistant from the line PB, and therefore PB is parallel to AC. We see similarly that PC is parallel to AB, and hence that $ABPC$ is a parallelogram. Now the triangles PBA and PBC also have the same perimeter, and $|AB| = |PC|$, hence $|AP| = |BC|$, i.e., the diagonals of the parallelogram $ABPC$ are equal. We deduce that $ABPC$ is a rectangle, and so triangle ABC has a right angle at vertex A.

   (b) Since the triangles PAB and PAC have the same perimeter, we have $|BA| + |BP| = |CA| + |CP| = r$, where r is some positive real number. We deduce that the vertices B and C lie on the locus of all points M in the plane of ABC such that $|MA| + |MP| = r$, which is an ellipse $\mathcal{E}$ with foci at A and P. In addition, since triangles PAB and PAC have the same area and share a common side AP, points B and C have the same distance from the axis AP of $\mathcal{E}$. Since the line AP is an axis of symmetry for $\mathcal{E}$, we deduce that B is the reflection of C about this line (the only other possibility, that BC passes through the center of the ellipse, must be excluded since the side BC does not intersect the line segment AP). It follows that triangles PAB and PAC are equal to each other, and in particular that $|AB| = |AC|$. In a similar manner, one shows that $|AB| = |BC|$, and hence that ABC is an equilateral triangle.