34th ANNUAL UNIVERSITY OF MARYLAND
HIGH SCHOOL MATHEMATICS COMPETITION
PART II
November 28, 2012, 1:00-3:00pm
NO CALCULATORS
2 hours

Do each of the 5 problems on a separate piece of paper.
Put your name and your school's name in the upper right corner of each page.
You must explain your work on a problem to receive credit.

1. (a) Suppose 101 Dalmatians chase 2012 squirrels. Each squirrel gets chased by at most one Dalmatian, and each Dalmatian chases at least one squirrel. Show that two Dalmatians chase the same number of squirrels.
(b) What is the largest number of Dalmatians that can chase 2012 squirrels in a way that each Dalmatian chases at least one squirrel and no two Dalmatians chase the same number of squirrels?

2. Lucy and Linus play the following game. They start by putting the integers 1, 2, 3, ..., 2012 in a hat. In each round of the game, Lucy and Linus each draw a number from the hat. If the two numbers are a and b, they throw away these numbers and put the number $|a-b|$ back into the hat. After 2011 rounds, there is only one number in the hat. If it is even, Lucy wins. If it is odd, Linus wins.
(a) Prove that there is a sequence of drawings that makes Lucy win.
(b) Prove that Lucy always wins.

3. Suppose x is a positive real number and x^{1990}, x^{2001}, and x^{2012} differ by integers. Prove that x is an integer.

4. Suppose that each point in three-dimensional space is colored with one of five colors and suppose that each color is used at least once. Prove that there is some plane that contains at least four of the colors.

5. Two circles, C_1 and C_2, are tangent at point A, with C_1 lying inside C_2 (and $C_1 \neq C_2$). The line through their centers intersects C_1 at B_1 and C_2 at B_2. A line L is drawn through A and it intersects C_1 at P_1 (with $P_1 \neq A$) and intersects C_2 at P_2 (with $P_2 \neq A$). The perpendicular from P_2 to the line B_1B_2 intersects the line B_1B_2 at F. Prove that if the line P_1F is tangent to C_1 then F is the midpoint of the line segment B_1B_2.