34th ANNUAL UNIVERSITY OF MARYLAND
HIGH SCHOOL MATHEMATICS COMPETITION
PART II Solutions

1. (a) Let d_1 be the smallest number of squirrels chased by a Dalmatian, d_2 the second smallest, etc. If no two squirrels chased the same number of squirrels, then

$$1 \le d_1 < d_2 < \cdots < d_{101},$$

so $d_i \ge i$ for each i. Therefore,

$$2012 = d_1 + d_2 + \cdots + d_{101} \ge 1 + 2 + \cdots + 101 = 5151,$$

which is a contradiction. Therefore, two Dalmatians must have chased the same number of squirrels.
(b) By the reasoning in (a), if there are n Dalmatians, then we must have

$$2012 \ge 1 + 2 + 3 + \cdots + n = n(n+1)/2.$$

This requires that $n \le 62$. The conditions can be satisfied with 62 Dalmatians by having the first 61 Dalmatians chase 1, 2, 3, $\cdots$, 61 squirrels, and then having the 62nd Dalmatian chase 121 squirrels/ Therefore, the largest possible number of Dalmatians is 62. (Cruella De Vil is trying to get the number down to this amount.)

2. (a) In the first round, they draw 1007 and 1, and put 1006 back in the hat. On the second round, they draw 1008 and 2 and put 1006 back in the hat. They continue this way through drawing 2012 and 1006, and putting 1006 back in the hat. The hat now consists of 1006 numbers, all of which are 1006. They then each draw 1006 and put 0 back in the hat, and do this 503 times. At this point, the hat contains 503 numbers, all of which are 0. The remaining 502 rounds consist of drawing out two 0's and putting in one 0's. At the end, there is one 0 in the hat. Since 0 is even, Lucy wins.
(b) Let S be the sum of the numbers in the hat. After Lucy and Linus draw a and b, the sum is $S - a - b$. Let's suppose $a \ge b$. Then they put in $a - b$, so the sum is $S - 2b$. Similarly, if $b \ge a$, the new sum is $S - 2a$. Therefore, the parity of the sum does not change after each round. At the beginning, there are 1006 even numbers and 1006 odd numbers in the hat, so the sum is even. Therefore, at the end of the game, the sum is even and Lucy wins.

3. By assumption, $x^{2012} - x^{1990}$ and $x^{2001} - x^{1990}$ are integers, so

$$\frac{x^{2012} - x^{1990}}{x^{2001} - x^{1990}} = \frac{x^{22} - 1}{x^{11} - 1} = x^{11} + 1$$

is rational. Therefore, x^{11} is rational, and

$$x^{1990} = \frac{x^{2001} - x^{1990}}{x^{11} - 1}$$

is rational since the numerator is an integer and the denominator is rational. Therefore, $x = (x^{11})^{181}/x^{1990}$ is rational. Let d be the denominator of x. Then x^{2001} has denominator d^{2001} and x^{1990} has denominator d^{1990}. If $d \ne 1$, the denominators of x^{2001} and x^{1990} are different, so their difference cannot be an integer. Therefore, the denominator must be 1, so x is an integer.

4. Choose points A, B, C, D, E that have 5 different colors. If any 4 are coplanar, we're done. Otherwise, proceed as follows: *Claim:* The line through E and one of the other 4 points intersects the plane determined by the remaining 3 points. *Proof:* Since A, B, C, D are not coplanar, they determine a tetrahedron T. The tetrahedron is the intersection of the four half-spaces consisting of the points on one side of each of the four planes ABC, ABD, ACD, BCD. If E lies inside T, then line DE intersects triangle ABC. If E lies outside T, then it lies on the opposite side of one of the four planes ABC, ABD, ACD, BCD. For definiteness, assume it is ABC. Then E and D lie on opposite sides of the plane ABC, so the line DE intersects ABC, say at F. This proves the claim.

   If F has the same color as D or E, then plane ABC contains 4 colors. Otherwise, F has the same color as one of A, B, C, so line DE contains three different colors. The plane containing the line DE and one point of a fourth color has four different colors.

5. Let M_1 be the center of C_1 and let M_2 be the center of C_2. Triangles AM_1P_1 and AM_2P_2 are isosceles, so let $\alpha = \angle M_1AP_1 = \angle M_1P_1A = \angle M_2P_2A$. Exterior angles are equal to the sums of the other two angles, so $\angle P_1M_1B_1 = \angle P_2M_2F = 2\alpha$. Therefore, M_1P_1 and M_2P_2 are parallel, which implies that P_2M_2 intersects FP_1 at I in a right angle. Since triangles P_2FM_2 and P_2IF are right angles sharing a common acute angle, they are similar, so $\angle P_2FI = \angle P_2M_2F = 2\alpha$. Let $\beta = \angle FP_1P_2$. Then $\beta + \alpha + 90° = 180°$, so $\beta = 90° - \alpha$. We now see that

$$\angle P_1P_2F = 180° - \angle P_1FP_2 - \angle FP_1P_2 = 180° - 2\alpha - \beta = 2\beta - \beta = \beta.$$

   Therefore, triangle P_2FP_1 is isosceles, which implies that $P_2F = P_1F$. Each of the right triangles FP_1M_1 and FP_2M_2 has an angle 2α, so they are similar, and they have congruent legs ($P_1F = P_2F$). Therefore, they are congruent, so $M_1F = M_2P_2 = r_2$, the radius of C_2. This implies that $FB_1 = r_2 - r_1$, where r_1 is the radius of C_1. Since $B_1B_2 = 2r_2 - 2r_1 = 2FB_1$, the midpoint of B_1B_2 is F.