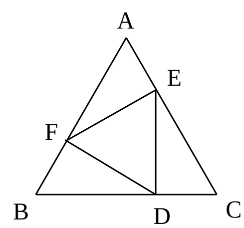
THE 35th ANNUAL (2013) UNIVERSITY OF MARYLAND

HIGH SCHOOL MATHEMATICS COMPETITION

PART II SOLUTIONS

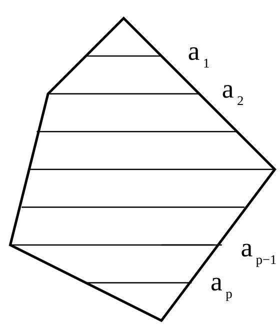
1. The sum of the integers in the first row of the array is $S := 1 + 2 + \cdots + 10 = 55$. The sum of the numbers in the second row is $2 + 4 + \cdots + 20 = 2(1 + 2 + \cdots + 10) = 2S$, and similarly the sum of the numbers in the k-th row is kS, for $1 \le k \le 10$. Therefore the required total sum is $S + 2S + \cdots + 10S = (1 + 2 + \cdots + 10)S = S^2 = 3025$.

2. Since angle $\angle DCE = 60^\circ$ in the right triangle CDE, we have $|CE| = 2|CD|$. Since triangle AEF is equal to triangle CDE, we have $|CD| = |AE|$. If $x = |AC|$, we deduce that $|CD| = |AC|/3 = x/3$ and $|DE| = |CD| \tan 60^\circ = x\sqrt{3}/3$. The area of ABC is given by the formula $x^2\sqrt{3}/4$, and the area of DEF is given by $x^2\sqrt{3}/12$, and hence equals $1/3$ of the area of ABC, which is known to be 1 square unit. The area of DEF is therefore equal to $1/3$ (in square units).



3. Suppose that S_n is a symmetric triangular set of points as in the figure, but with n points in the bottom row. We claim that if a collection of m lines has the property that for every point in the set, there is at least one line in the collection that passes through that point, then $m \ge n$. We prove the claim by induction on n. The claim is clearly true when $n = 1$. Assume that the claim holds for the set S_{n-1}, and we will prove that it holds for S_n. Consider a collection of m lines with the stated property. If one of the lines passes through two points in the bottom row of S_n, then it passes through all the n points in the bottom row. Since removal of this row from S_n results in the set S_{n-1}, which is covered by the remaining $m - 1$ lines, the inductive hypothesis gives $m - 1 \ge n - 1$, and hence $m \ge n$. If no lines in the collection pass through more than one point in the bottom row, then clearly $m \ge n$ since each of the n points in this row must be contained in at least one line in the collection. This completes the induction, and the proof.

4. The horizontal grid lines contained in the interior of P divide the polygon into a union of two triangles and a number of trapezoids with mutually disjoint interiors.



If the lengths of these horizontal grid lines are $a_1, \ldots, a_p$ going from top to bottom as in the figure, then the two triangles at the ends have areas $a_1/2$ and $a_p/2$, while the k-th trapezoid has area $(a_k + a_{k+1})/2$, for $1 \le k \le p - 1$. The sum of these areas is

$$\frac{a_1}{2} + \frac{a_1 + a_2}{2} + \frac{a_2 + a_3}{2} + \cdots + \frac{a_{p-1} + a_p}{2} + \frac{a_p}{2} = a_1 + a_2 + \cdots + a_p = H.$$

We deduce that H is equal to the area of P. A similar argument proves that V is equal to the area of P as well. Therefore, we have $H = V$.

5. We will prove that the game is fair exactly when k is *not* a multiple of 3. The set $X = \{1, 2, \ldots, 2013\}$ is a union of the 671 three element subsets $S_1 = \{1, 2, 3\}, S_2 = \{4, 5, 6\}, \ldots, S_{671} = \{2011, 2012, 2013\}$. Let A be any subset of X with k elements, and let $[A]$ denote the remainder when we divide the sum of all the numbers in A by 3.

   If k is not divisible by 3, then there must be an index i with $A \cap S_i$ having 1 or 2 elements. Let i_o be the smallest such index, and define a function $f : X \to X$ by $f(n) = n$, if $n \notin S_{i_0}$, $f(n) = n + 1$, if $n \in S_{i_0}$ and $n+1 \in S_{i_0}$, and $f(n) = n - 2$, if $n \in S_{i_0}$ and $n+1 \notin S_{i_0}$. The function f is a bijection cyclically permutes the elements of S_{i_0}, and leaves all the other elements of X fixed. Let $f(A)$ be the k-element subset of X obtained by applying f to every element in A. Then $[A]$, $[f(A)]$, and $[f(f(A))]$ are distinct remainders, and $f(f(f(A))) = A$. We deduce that the function f gives a 1-1 correspondence between those k-element subsets A of X with $[A] = 0$ (when Peter wins), those with $[A] = 1$ (when Paul wins), and those with $[A] = 2$ (when Mary wins). The game is therefore fair.

   Now suppose that $k = 3m$ is a multiple of 3. Then the k-element subsets A such that $A \cap S_i$ has 1 or 2 elements for some i can be partitioned into triples using the same function f as above. When A is chosen at random among these subsets, the result is an equal number of winning games for each of the three players. There remain those k-element subsets A which are unions of m 3-element subsets S_i. Since all such subsets satisfy $[A] = 0$, Peter will win if A is chosen among those subsets. We deduce that the game is biased towards a win for Peter.