THE 36$^{\text{th}}$ ANNUAL (2014) UNIVERSITY OF MARYLAND

HIGH SCHOOL MATHEMATICS COMPETITION

PART II SOLUTIONS

1. (a) The prime number 7 can only appear in one row of the array, and the product of the numbers in that row will be a multiple of 7. Since the product of the three numbers in each of the other two rows will not be a multiple of 7, the numbers 1, 2, 3, ..., 9 cannot be used to form a multimagic square.

   (b) Any 3×3 magic square M gives rise to a multimagic square by replacing each entry x in M with 2^x. For example, the magic square $\begin{matrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{matrix}$ gives rise to the multimagic square $\begin{matrix} 2^4 & 2^9 & 2^2 \\ 2^3 & 2^5 & 2^7 \\ 2^8 & 2^1 & 2^6 \end{matrix}$.

2. Let $n_k := (2014!)/k$ for $1 \le k \le 2014$. Then the difference

$$\frac{1}{n_{k+1}} - \frac{1}{n_k} = \frac{k+1}{2014!} - \frac{k}{2014!} = \frac{1}{2014!}$$

   for $1 \le k \le 2013$, and therefore is independent of k. It follows that $\frac{1}{n_1}, \frac{1}{n_2}, \ldots, \frac{1}{n_{2014}}$ is an arithmetic progression.

3. Write $x = n + \epsilon$ where n is an integer and $0 \le \epsilon < 1$. Then we seek the pairs (n, ϵ) such that $(n+\epsilon)^2 - 25n + 100 = 0$, or equivalently,

$$\epsilon^2 + 2n\epsilon + (n^2 - 25n + 100) = 0. \tag{1}$$

   Solving the quadratic equation (1) for $\epsilon \ge 0$ gives $\epsilon = -n + 5\sqrt{n-4}$. The pair (n, ϵ) will give a solution x to the original equation if and only if

$$0 \le -n + 5\sqrt{n-4} < 1. \tag{2}$$

   We have

$$0 \le -n + 5\sqrt{n-4} \iff n^2 - 25n + 100 \le 0 \iff (n-5)(n-20) \le 0,$$

   which gives $5 \le n \le 20$. The inequality $-n + 5\sqrt{n-4} < 1$ is equivalent to $25(n-4) < (1+n)^2$, or $n^2 - 23n + 101 \ge 0$. The roots of the quadratic equation $y^2 - 23y + 101 = 0$ are $(23 \pm 5\sqrt{5})/2$, which are approximately 5.9 and 17.1. It follows that n lies outside the interval $[5.9, 17.1]$. Since we also know that $n \in [5, 20]$, we conclude that the only integers n for which (2) holds are $n = 5$ and $n = 18, 19$, and 20. As $x^2 = \sqrt{25n - 100}$, we correspondingly obtain the four solutions of the original equation: $x = 5$, $x = \sqrt{350}$, $x = \sqrt{375}$, and $x = 20$.

4. Split the carts into four groups of two carts each: C_1, C_2, C_3, and C_4. Assume without loss of generality that the desert is four miles across. For $1 \le k \le 4$, the carts in group C_k carry 2 cannons k miles ahead across the desert, leave them there, move backwards 1 mile, pick up the two cannons which they find there, and carry them forwards $5-k$ miles to the end of the desert. This accomplishes the task in one and a half days.

   We next prove that one and a half days is the minimum possible amount of time required. Suppose that the 8 carts move the 10 cannons across the desert in t days. For each i with $1 \le i \le 8$, let t_i be the time cart i was moving a cannon forward during the process, and $t_i' := t - t_i$ be the time when it was either standing still or moving backwards. Since there are 2 more cannons than carts, we must have at least 2 days worth of cart time going backwards to retrieve them, which gives the inequality $\sum_{i=1}^{8} t_i' \geq 2$. It follows that

$$8t = \sum_{i=1}^{8}(t_i + t_i') = \sum_{i=1}^{8} t_i + \sum_{i=1}^{8} t_i' \geq 10 + 2 = 12, \tag{3}$$

which implies that $t \geq \frac{3}{2}$, as required.

5. Let us consider the general case where the polygon C has n sides and $n(n-3)/2$ diagonals. Suppose that the vertices of C are $A_0, A_1, \ldots, A_{n-1}$, in clockwise order. If A_rA_s is any diagonal of C, then A_rA_s intersects $A_{r+1}A_{s+1}$ at a point M in the interior of C (here, we take the indices modulo n, that is, we subtract n if necessary to get integers in $[0, n-1]$). The triangle inequality in the triangles A_rMA_{r+1} and A_sMA_{s+1} gives $|MA_r| + |MA_{r+1}| > |A_rA_{r+1}|$ and $|MA_s| + |MA_{s+1}| > |A_sA_{s+1}|$. Adding these two inequalities together gives

$$|A_rA_s| + |A_{r+1}A_{s+1}| > |A_rA_{r+1}| + |A_sA_{s+1}|. \tag{4}$$

Now consider the sum of the inequalities (4) over all diagonals A_rA_s of C. Each diagonal of C will occur twice on the left hand side of this sum. Since each vertex A_r is an endpoint of $n-3$ diagonals, each side of C will appear $n-3$ times on the right hand side. We conclude that $2d > (n-3)p$, or $\frac{d}{p} > \frac{n-3}{2}$. Applying this with $n = 4031$ gives $\frac{d}{p} > \frac{4028}{2} = 2014$.