

37th ANNUAL UNIVERSITY OF MARYLAND
HIGH SCHOOL MATHEMATICS COMPETITION

PART II

December 2, 2015, 1:00–3:00

NO CALCULATORS

2 hours

- Nine coins are placed in a row, alternating between heads and tails as follows: H T H T H T H T H. A legal move consists of turning over any two adjacent coins.
 - Give a sequence of legal moves that changes the configuration into H H H H H H H H H.
 - Prove that there is no sequence of legal moves that changes the original configuration into T T T T T T T T T.

- Find (with proof) all integers k that satisfy the equation

$$\begin{aligned} & \frac{k-15}{2000} + \frac{k-12}{2003} + \frac{k-9}{2006} + \frac{k-6}{2009} + \frac{k-3}{2012} \\ &= \frac{k-2000}{15} + \frac{k-2003}{12} + \frac{k-2006}{9} + \frac{k-2009}{6} + \frac{k-2012}{3}. \end{aligned}$$

- Some (not necessarily distinct) natural numbers from 1 to 2015 are written on 2015 lottery tickets, with exactly one number written on each ticket. It is known that the sum of the numbers on any nonempty subset of tickets (including the set of all tickets) is not divisible by 2016. Prove that the same number is written on all of the tickets.
- A set of points A is called *distance-distinct* if every pair of points in A has a different distance.
 - Show that for all infinite sets of points B on the real line, there exists an infinite distance-distinct set A contained in B .
 - Show that for all infinite sets of points B on the real plane, there exists an infinite distance-distinct set A contained in B .
- Let $ABCD$ be a (not necessarily regular) tetrahedron and consider six points E, F, G, H, I, J on its edges AB, BC, AC, AD, BD, CD , respectively, such that

$$|AE| \cdot |EB| = |BF| \cdot |FC| = |AG| \cdot |GC| = |AH| \cdot |HD| = |BI| \cdot |ID| = |CJ| \cdot |JD|.$$

Prove that the points E, F, G, H, I , and J lie on the surface of a sphere.