THE 37th ANNUAL (2015) UNIVERSITY OF MARYLAND

HIGH SCHOOL MATHEMATICS COMPETITION

PART II SOLUTIONS

1. (a) One possible such sequence is H T H T H T H T H $\mapsto$ H H T T H T H T H $\mapsto$ H H H H H T H T H $\mapsto$ H H H H H H T T H $\mapsto$ H H H H H H H H H.

   (b) After any legal move, the total number of T's in the configuration is either decreased by 2, increased by 2, or remains unchanged. Since there are 4 T's in the initial configuration, we deduce that after any sequence of legal moves the total number of T's in the resulting configuration will be even. But there are 9 T's in T T T T T T T T T; hence this configuration cannot be achieved.

2. We observe by inspection that $k = 2015$ is a solution to the equation. Also, the equation is linear in k, i.e., it may be written in the form $ak = b$, where

$$a = \left(\frac{1}{15} + \frac{1}{12} + \frac{1}{9} + \frac{1}{6} + \frac{1}{3}\right) - \left(\frac{1}{2000} + \frac{1}{2003} + \frac{1}{2006} + \frac{1}{2009} + \frac{1}{2012}\right) > \frac{1}{3} - \frac{5}{2000} > 0.$$

   Since $a \neq 0$, we deduce that the equation $ak = b$ has a unique solution $k = b/a$, and therefore that the answer $k = 2015$ is the only one.

3. Let x_i be the number written on the i-th ticket for $1 \leq i \leq 2015$. It suffices to show that $x_1 = x_2$, for then by reordering the tickets and using the same argument we can prove that $x_i = x_j$ for all i and j. Consider the partial sums $s_i := x_1 + \cdots + x_i$, and let r_i be the remainder when s_i is divided by 2016. The assumptions give $r_i \neq 0$ for all $i \in [1, 2015]$. If $r_i = r_j$ for some $i < j$, then 2016 divides $s_j - s_i = x_{i+1} + \cdots + x_j$, which is not allowed. Therefore the remainders $r_1 = x_1, r_2, \ldots, r_{2015}$ are equal to some permutation of the numbers $1, \ldots, 2015$.

   By switching the first two tickets, we form a new sequence $x_2, x_1, x_3, \ldots, x_{2015}$ with partial sums $s'_1 = x_2$ and $s'_i = s_i$ for all $i \geq 2$. The remainders of the s'_i when divided by 2016 are $r'_1 = x_2, r'_2 = r_2, \ldots, r'_{2015} = r_{2015}$, and these also must form a permutation of the numbers $1, \ldots, 2015$. By comparing with the previous sequence of remainders, we deduce that $x_1 = x_2$, as required. Therefore, the same number is written on all of the tickets.

4. (a) We will recursively construct a sequence of distance-distinct sets $A_1, A_2, \ldots$ contained in B such that $A_1 \subset A_2 \subset \cdots$ and A_i consists of i points for every $i \geq 1$. Let A_1 be any point of B. Assume by induction on n that a distance-distinct set $A_n \subset B$ with n points has been constructed, and let D_n be the finite set of distances between any two points of A_n. Let M' be the set of midpoints of the line segments which connect any two points of A_n, and let M'' be the set of all points on the real line whose distance from some point $P \in A_n$ lies in D_n. Then both M' and M'' are finite sets. Choose a point Q in B that does not belong to the set $A_n \cup M' \cup M''$. Then the construction ensures that $A_{n+1} := A_n \cup \{Q\}$ is distance distinct.

   Define A to the be the union of the sets A_n for all $n \geq 1$. Then A is an infinite subset of B which is distance distinct. Indeed, if P_1, P_2, P_3, and P_4 are any four points in A, then there exists an index n such that all four points are in A_n. Since A_n is distance distinct, we conclude that $|P_1P_2| \neq |P_3P_4|$.

   (b) If there is some line L in the plane which contains infinitely many points of B, then by part (a) there is an infinite distance-distinct subset of $B \cap L$. A similar argument also implies that if there is some circle C in the plane which contains infinitely many points of B, then there is an infinite distance-distinct subset of $B \cap C$ (the only change to the proof in (a) is that M' should be replaced with the set of midpoints of all arcs of C which connect any two points of A_n). We may therefore assume that the intersection of B with any line and any circle is finite.

We can now work similarly to part (a), to construct a sequence of distance-distinct sets $A_1, A_2, \ldots$ contained in B such that $A_1 \subset A_2 \subset \cdots$ and A_i consists of i points for each i. Let A_1 be any point of B. Assume inductively that a distance-distinct set $A_n \subset B$ with n points has been constructed, and let D_n be the finite set of distances between any two points of A_n. Let M' be the set of perpendicular bisectors of all line segments which connect two points of A_n, and let M'' be the set of all circles whose centers lie in A_n and radii lie in D_n. Then both $B \cap M'$ and $B \cap M''$ are finite sets. Since B is infinite, there is a point Q in B that does not belong to the set $A_n \cup M \cup M'$. Then the construction ensures that $A_{n+1} := A_n \cup \{Q\}$ is distance distinct. Finally, let A be the union of the sets A_n for all $n \geq 1$. As in part (a), A is a solution to the problem.

5. First Solution: Let $\mathcal{C}$ be a circle with center O and radius R, and let P be any point inside $\mathcal{C}$. If AB is a chord of $\mathcal{C}$ passing through P, then the quantity $d(P) := |PA| \cdot |PB|$ is called the power of the point P with respect to $\mathcal{C}$, and is independent of the chord AB chosen. Moreover, we have $d(P) = R^2 - |OP|^2$, as is easily seen by choosing AB to be a diameter of the circle.

   Let $\mathcal{C}_1$ be the circumcircle of triangle ABC, with center O_1. By our assumption, the points E, F, and G have the same power with respect to $\mathcal{C}_1$, and therefore satisfy $|EO_1| = |FO_1| = |GO_1|$. If O is the center of the circumsphere of $ABCD$, then O lies on the line through O_1 which is perpendicular to the plane ABC. It follows that the right triangles EOO_1, FOO_1, and GOO_1 are equal, and hence that $|EO| = |FO| = |GO|$. By considering triangles ABD and BCD, we similarly prove that $|EO| = |HO| = |IO|$ and $|FO| = |HO| = |JO|$. We conclude that the six points E, F, G, H, I, and J are equidistant from O, and thus lie on a sphere with center at O.

   Second Solution: Let $\mathcal{S}$ be a sphere with center O and radius R, and let P be any point inside $\mathcal{S}$. If AB is a chord of $\mathcal{S}$ passing through P, we claim that the quantity $d(P) := |PA| \cdot |PB|$ is independent of the chord AB chosen, and is equal to $R^2 - |OP|^2$. Indeed, the point O and the chord AB lie on a plane which intersects $\mathcal{S}$ in a great circle $\mathcal{C}$, and $d(P)$ is equal to the power of the point P with respect to $\mathcal{C}$. This implies that $d(P) = R^2 - |OP|^2$ and hence is independent of the chord AB chosen. The quantity $d(P)$ is called the power of P with respect to the sphere $\mathcal{S}$.

   In the case of the problem at hand, the assumption implies that the six points E, F, G, H, I, and J have the same power with respect to the circumsphere $\mathcal{S}$ of $ABCD$ (the unique sphere that passes through the 4 points A, B, C, and D). It follows that E, F, G, H, I, and J are equidistant from the center O of $\mathcal{S}$, and therefore lie on a smaller sphere with the same center O.