THE 38th ANNUAL (2016) UNIVERSITY OF MARYLAND

HIGH SCHOOL MATHEMATICS COMPETITION

PART II SOLUTIONS

1. The only such table is

2	1	5	7	70
1	3	7	7	147
8	8	8	1	512
9	9	1	9	729
144	216	280	441	

2. Suppose X is a pdf. Note that if $x \in X$, then $x+2, x+3, x+5, x+7 \notin X$. Also, from $x+1, x+4$ and $x+6$ at most one can be in X. Thus, among every eight consecutive integers at most 2 may be in X. This implies X has at most $\dfrac{2016}{4} = 504$ elements.

   The set $X = \{4k \mid 1 \leq k \leq 504\}$ is a pdf, because if $x, y \in X$, then $|x-y|$ is a multiple of 4. Therefore $|x-y|$ is not prime.

   The answer is 504.

3. For $i \in \{1, 2, \ldots, 14\}$, let M_i denote the midpoint of the line segment X_iX_{i+1}. Suppose, for the sake of contradiction, that each of these points is within distance $1/2$ from the origin. Then, by the triangle inequality the distance between any two points M_i and M_j is no more than 1 for all i, j.

   Note that $M_1X_2 = X_1X_2/2$, $M_2X_2 = X_3X_2/2$, and $\angle M_1X_2M_2 = \angle X_1X_2X_3$, and thus $\triangle X_1X_2X_3$ is similar to $\triangle M_1X_2M_2$ and $M_1M_2 = X_1X_3/2$. We have $X_1X_3 = 2M_1M_2 \leq 2 \times 1 = 2$. Repeating this reasoning, we find that the distances $X_{2k-1}X_{2k+1}$ are all no more than 2, for $k \in \{1, 2, \ldots, 7\}$. By the triangle inequality, the distance from X_1 to X_{15} is less than or equal to $7 \cdot 2 = 14$. But it was assumed that $X_1 = (10, 0)$ and $X_{15} = (0, 10)$, and so the distance X_1X_{15} is actually $\sqrt{10^2 + 10^2} = 10\sqrt{2}$. Since $14^2 = 196 < 200 = (10\sqrt{2})^2$, we find that the distance X_1X_{15} exceeds 14, which is a contradiction. This completes the proof.

4. For every $k \in \{1, 2, \ldots, 82\}$, there is an associated three-letter sequence (s_k, s_{k+1}, s_{k+2}). There are 27 distinct three-letter sequences that can be constructed from $\{A, B, C\}$. Since $82 = 27 \cdot 3 + 1$, the pigeonhole principle implies that some three-letter sequence (r, s, t) must appear more than 3 times as a consecutive subsequence of $(s_1, \ldots, s_{84})$. One such occurrence must be at the end of $(s_1, s_2, ..., s_{84})$, since otherwise one of the three sequences $(r, s, t, A), (r, s, t, B), (r, s, t, C)$ would have to appear more than once as a consecutive subsequence of $(s_1, s_2, ..., s_{84})$. Similarly, one such occurrence must be at the beginning of $(s_1, s_2, ..., s_{84})$, since otherwise one of the sequences $(A, r, s, t), (B, r, s, t), (C, r, s, t)$ would have to appear appear more than once as a consecutive subsequence of $(s_1, s_2, ..., s_{84})$. Therefore, $(s_{82}, s_{83}, s_{84}) = (r, s, t) = (s_1, s_2, s_3) = (A, B, B)$. The correct answer is B.

   Remark. Although it was not required for credit, here is an example of a sequence $(s_1, s_2, ..., s_{84})$ satisfying the conditions of the problem. (This example comes from the multiplicative structure of the finite field of order 81.)

$$
\begin{gathered}
(A, B, B, C, A, B, C, B, A, C, C, A, B, A, B, A, A, A, A, C, C, C, B, A, A, C, A, C, \\
B, B, B, C, B, B, C, C, B, C, A, C, C, B, B, A, C, B, A, B, C, A, A, C, B, C, B, C, \\
C, C, C, A, A, A, B, C, C, A, C, A, B, B, B, B, A, B, B, A, A, B, A, C, A, A, B, B)
\end{gathered}
$$

5. We claim that there is no such sequence. On the contrary assume a_n is such a sequence. For any positive integer n, we use the AM-GM inequality to obtain

$$(\sum_{k=n+1}^{2n} a_k)(\sum_{k=n+1}^{2n} \frac{1}{a_k}) \geq n(\prod_{k=n+1}^{2n} a_k)^{1/n} \cdot n(\prod_{k=n+1}^{2n} \frac{1}{a_k})^{1/n} = n^2$$

On the other hand,

$$\sum_{k=n+1}^{2n} a_k \leq \sum_{k=1}^{2n} a_k \leq (2n)^2 = 4n^2$$

Combining these two inequalities we obtain

$$\sum_{k=n+1}^{2n} \frac{1}{a_k} \geq \frac{1}{4}$$

Adding up this inequality for $n = 2, 4, 8, \ldots, 2^m$, we obtain

$$\sum_{k=1}^{2^{m+1}} \frac{1}{a_k} \geq \frac{m}{4}$$

This is larger than 2016 when $m > 4 \cdot 2016$, which is a contradiction.