39th ANNUAL UNIVERSITY OF MARYLAND HIGH SCHOOL MATHEMATICS COMPETITION PART II

November 29, 2017, 1:00–3:00

NO CALCULATORS
2 hours

1. Consider the following four statements referring to themselves:

   1. At least one of these statements is true.
   2. At least two of these statements are false.
   3. At least three of these statements are true.
   4. All four of these statements are false.

   Determine which statements are true and which are false. Justify your answer.

2. Let

   $$f(x) = a_{2017}x^{2017} + a_{2016}x^{2016} + \cdots + a_1 x + a_0$$

   where the coefficients $a_0, a_1, \ldots, a_{2017}$ have not yet been determined. Alice and Bob play the following game:

   - Alice and Bob alternate choosing nonzero integer values for the coefficients, with Alice going first. (For example, Alice's first move could be to set a_{18} to -3.)
   - After all of the coefficients have been chosen:
     - If $f(x)$ has an integer root then Alice wins.
     - If $f(x)$ does not have an integer root then Bob wins.

   Determine which player has a winning strategy and what the strategy is. Make sure to justify your answer.

3. Suppose that a circle can be inscribed in a polygon P with 2017 equal sides. Prove that P is a regular polygon; that is, all angles of P are also equal.

4. A $3 \times 3 \times 3$ cube of cheese is sliced into twenty-seven $1 \times 1 \times 1$ blocks. A mouse starts anywhere on the outside and eats one of the $1 \times 1 \times 1$ cubes. He then moves to an adjacent cube (in any direction), eats that cube, and continues until he has eaten all 27 cubes. (Two cubes are considered adjacent if they share a face.) Prove that no matter what strategy the mouse uses, he cannot eat the middle cube last.

   [**Note:** One should neglect gravity – intermediate configurations don't collapse.]

5. Suppose that a constant $c > 0$ and an infinite sequence of real numbers $x_0, x_1, x_2, \ldots$ satisfy

   $$x_{k+1} = \frac{x_k + 1}{1 - cx_k}$$

   for all $k \geq 0$. Prove that the sequence $x_0, x_1, x_2, \ldots$ contains infinitely many positive terms and also contains infinitely many negative terms.