THE 39th ANNUAL (2017) UNIVERSITY OF MARYLAND

HIGH SCHOOL MATHEMATICS COMPETITION

PART II SOLUTIONS

1. If statement 4 is true, then all four statements must be false. Therefore statement 4 cannot be true. Thus statement 4 is false. If statement 3 is true, then 1 and 2 must be true. 2 implies at most 2 of the statements are true, so 3 is false. Thus, 2 is true and so is 1.

2. Bob has a winning strategy. In his first move, Bob makes sure a_0 is selected (if Alice has not selected it yet.) By the Integer Root Theorem the only integer roots of the polynomial are divisors of a_0. This leaves finitely many possible values for integer roots. In his last move, Bob selects the last coefficient so that none of the divisors of a_0 is a root.

3. Let O be the center of the inscribed circle and $A_1, A_2, \ldots, A_{2017}$ be consecutive vertices of the polygon. Note that since the circle is tangent to all sides of P, O lies on all internal angle bisectors of P. Let H and K be the points of tangency of the circle with sides A_1A_2 and A_2A_3, respectively. We know $|HA_2| = |A_2K|$. By assumption $|A_1H| = |A_3K|$. Therefore triangles A_1HO and A_3KO are congruent, by SAS. This implies $\angle OA_1H = \angle OA_3K$. Therefore the internal angles A_1 and A_3 are equal. Repeating this we conclude,

$$A_1 = A_3 = \cdots = A_{2015} = A_{2017} = A_2 = A_4 = \cdots = A_{2016}$$

4. Color the cubes white or black in a way that every two adjacent cubes have different colors. (This is essentially a 3D-checkerboard coloring.) Suppose corner cubes are black. Thus, we will have 14 black and 13 white cubes. Note that every path alternates between white and black cubes and the center cube is white. Since we have more black cubes than white, a path cannot end with a white cube. Thus, the center cube cannot be the last cube in the path.

5. Suppose on the contrary $x_k \geq 0$ for all $k \geq N$. Thus, $x_{k+1} \geq 0$ for all $k \geq N$. This implies $1 - cx_k \geq 0$. Therefore $x_k \leq 1/c$. On the other hand $x_{k+1} - x_k = \dfrac{1 + cx_k^2}{1 - cx_k} \geq 1$. Therefore $x_{k+1} \geq x_k + 1$, which implies $x_{N+m} \geq x_N + m$, for all $m \geq 0$. This contradicts $x_k \leq 1/c$.

   If $x_k \leq 0$ for all $k \geq N$, then $x_{k+1} \leq 0$, which implies $1 + x_k \leq 0$. This implies $x_{k+1} - x_k \geq 1$, which implies $x_{N+m} \geq x_N + m$, for all $m \geq 0$. This contradicts $x_k \leq 0$.