THE 40^{th} ANNUAL (2018) UNIVERSITY OF MARYLAND

HIGH SCHOOL MATHEMATICS COMPETITION

PART II SOLUTIONS

1. Suppose the contents of the envelopes in dollars are $x, x+1, \ldots, x+5$. The total money in the envelopes, thus, is $6x+15$. The total amount in the remaining envelopes is between $x+x+1+\ldots+x+4 = 5x+10$ and $x+1+\ldots+x+5 = 5x+15$. Thus, $5x+10 \leq 2018 \leq 5x+15$, which implies $x+2 \leq 2018/5 \leq x+3$. This shows $x = 401$. Since $2018 = 5 \times 401 + 13$, the last envelope contains $x+2 = 403$ dollars. The answer is 403.

2. Let O be the center of the circle. Note that A and O are both on the perpendicular bisector of BC. Thus $AO \perp BC$. Since DE and BC are parallel and $AO \perp BC$, we have $AO \perp DE$.

   Let r be the radius of the circle and M be the point of intersection of DE and AO. We need to show $|MO| > r$. Since triangles AOB and ADM are similar, we have

$$\frac{|AO|}{|AD|} = \frac{|AB|}{|AM|}$$

   Since $|AD| = |AB|/2$, we have

$$|AM| = \frac{|AB|^2}{2|AO|} = \frac{|AO|^2 - r^2}{2|AO|}$$

   This shows $|MO| = |AO| - |AM| = \dfrac{|AO|^2 + r^2}{2|AO|}$. To show $|MO| > r$, it is enough to show $|AO|^2 + r^2 > 2r|AO|$, which is equivalent to $(|AO| - r)^2 > 0$. This holds since $|AO| > r$.

3. Let $T = \{x+1 \mid x \in S\} \subseteq \{1, \ldots, n-1\}$. By assumption, S is closed if and only if T satisfies the following

   (a) $1 \in T$.

   (b) If $x \in T$, then $n - x \in T$.

   (c) If $x \in T$, $y \geq 1$ and y divides x, then $y \in T$.

   Suppose n is prime and S is closed. We will show $T = \{1, \ldots, n-1\}$. On the contrary assume k is the smallest positive integer less than or equal to $n-1$ that is not in T. Note that $k \geq 2$ and thus k does not divide n. By division algorithm, there are integers q and r for which $n = kq + r$ and $0 < r < k$. By the choice of k, we have $r \in T$ and thus by (b), $n - r \in T$. This implies $kq \in T$. By (c), $k \in T$, which is a contradiction. This shows $T = \{1, \ldots, n-1\}$.

   Now assume n is composite and let T be the set of all positive integers less than or equal to $n-1$ that are relatively prime to n. Clearly (a) is satisfied. If $x \in T$, then $n - x$ is also relatively prime to n and thus $n - x \in T$, which shows (b) is satisfied. If x is relatively prime to n, then every divisor of x is also relatively prime to n, which shows (c) is satisfied. Also, if $1 < r < n$ is a divisor of n, then $r \notin T$, which means $T \neq \{1, \ldots, n-1\}$. This completes the proof.

4. For any n, let a_n, c_n, and i_n denote the total number of knight paths of length n which begin at A and end at A, C, and I, respectively. We are trying to find $a_{2018} - i_{2018}$.

   First we evaluate a_2 and i_2. There are two knight paths from A to A (specifically, AHA and AFA), one path from A to C (AHC), one path from A to G (AFG) and no paths from A to I. Thus $a_2 = 2$ and $i_2 = 0$.

   Since there are 2 knight paths of length 2 from A to A and one each from C to A and G to A, we have $a_{n+2} = 2a_n + c_n + c_n = 2a_n + 2c_n$

By similar reasoning, $i_{n+2} = 2i_n + c_n + c_n = 2i_n + 2c_n$.

Subtracting these two equations we obtain $a_{n+2} - i_{n+2} = 2(a_n - i_n)$. Therefore, $a_{2018} - i_{2018} = 2^{1008}(a_2 - i_2) = 2^{1009}$.

5. We claim this is impossible. Suppose on the contrary n strips of widths $d_1, \ldots, d_n$ cover a unit disk D. Let S be the sphere whose great circle is the boundary of D. For each strip draw two planes perpendicular to the plane containing D through parallel lines that bound that strip. Let $R_1, \ldots, R_n$ be the regions that are created by these planes on the sphere S. Note that by assumption the sum of areas of $R_1, \ldots, R_n$ is at least the area of S.

   Recall that the area of a spherical cap is $2\pi rh$, where r is the radius of the sphere and h is the height of the spherical cap. Since the area of each ring R_i is the difference of the areas of two spherical caps, the area of R_i is $2\pi \times \frac{1}{2} \times d_i$. This implies $\pi d_1 + \cdots + \pi d_n \geq 4\pi \frac{1}{4} = \pi$. Therefore, $d_1 + \cdots + d_n \geq 1$, which is a contradiction.