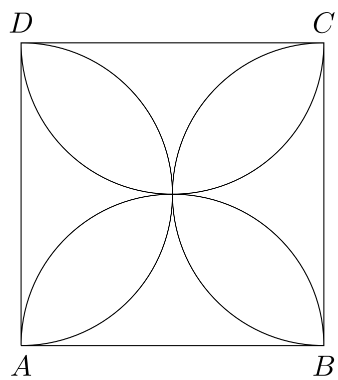
41st ANNUAL UNIVERSITY OF MARYLAND
HIGH SCHOOL MATHEMATICS COMPETITION
PART II
November 20, 2019, 1:00–3:00 pm

Instructions: Solutions to different problems should go on separate pages. Write down your name and the problem number on the upper left corner of all pages that you submit for grading. Show your work and justify all of your steps. Submit only what you want to be graded; no blank paper or scratch paper.

NO CALCULATORS
2 hours

1. Alex and Sam have a friend Pat, who is younger than they are. Alex, Sam and Pat all share a birthday. When Pat was born, Alex's age times Sam's age was 42. Now Pat's age is 33 and Alex's age is a prime number. How old is Sam now? Show your work and justify your answer. (All ages are whole numbers.)

2. Let $ABCD$ be a square with side length 2. The four sides of $ABCD$ are diameters of four semicircles, each of which lies inside the square. The set of all points which lie on or inside two of these semicircles is a four petaled flower. Find (with proof) the area of this flower.



3. A prime number is called *strongly prime* if every integer obtained by permuting its digits is also prime. For example 113 is strongly prime, since 113, 131, and 311 are all prime numbers. Prove that there is no strongly prime number such that each of the digits 1, 3, 7, and 9 appears at least once in its decimal representation.

4. Suppose n is a positive integer. Let a_n be the number of permutations of $1, 2, \ldots, n$, where i is not in the i-th position, for all i with $1 \leq i \leq n$. For example $a_3 = 2$, where the two permutations that are counted are 231, and 312. Let b_n be the number of permutations of $1, 2, \ldots, n$, where no i is followed by $i+1$, for all i with $1 \leq i \leq n-1$. For example $b_3 = 3$, where the three permutations that are counted are $132, 213$, and 321. For every $n \geq 1$, find (with proof) a simple formula for $\frac{a_{n+1}}{b_n}$. Your formula should not involve summations. Use your formula to evaluate $\frac{a_{2020}}{b_{2019}}$.

5. Let $n \geq 2$ be an integer and $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 + a_2 + \cdots + a_n = 1$. Prove that

$$\sum_{k=1}^{n} \frac{a_k}{1 + a_{k+1} - a_{k-1}} \geq 1$$

(Here $a_0 = a_n$ and $a_{n+1} = a_1$.)