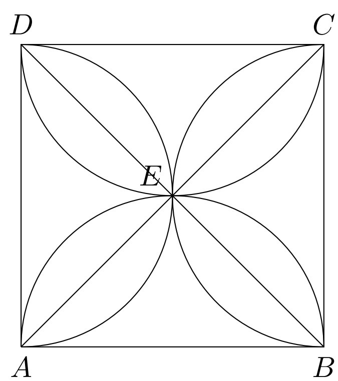
THE 41^{st} ANNUAL (2019) UNIVERSITY OF MARYLAND

HIGH SCHOOL MATHEMATICS COMPETITION

PART II SOLUTIONS

1. Let A and S be the ages of Alex and Sam when Pat was born. We know that $AS = 42$ and $A + 33$ is prime. This means, A must be even. Also, since 33 is divisible by 3, A cannot be divisible by 3. Therefore, $A = 2$, or 14. These give us $A + 33 = 35$, which is not prime or $A + 33 = 47$, which is prime. Therefore, $A = 14$, and thus $S = 3$. This means Sam's age now is 36.

2. Let E be the intersection of the diagonals of $ABCD$. Since AC and BD are perpendicular, E lies on all four circles.



The area of one of the leaves is equal to the area of one of the semicircles minus the area of triangle ABE. The area of each semicircle is $\pi/2$. The area of right triangle ABE is $2 \times 1/2 = 1$. Thus, the area of each leaf is $\dfrac{\pi}{2} - 1$. Therefore, the answer is $2\pi - 4$.

3. Suppose to the contrary that n is a strongly prime integer such that each of the digits 1, 3, 7, or 9 appears in its decimal representation. We observe that the remainders of 1379, 3179, 3719, 1739, 3791, 9371, 7139 when divided by 7, are 0, 1, 2, 3, 4, 5, and 6, respectively. This implies for every integer x, one of the integers $x+1379, x+3179, x+3719, x+1739, x+3791, x+9371, x+7139$ is divisible by 7.

   Suppose m is an integer obtained from all digits of n except for one of each 1, 3, 7, and 9. Then, by what we discussed above, one of the integers $m1379, m3179, m3719, m1739, m3791, m9371, m7139$ is divisible by 7, which is a contradiction.

   Note: It has also been shown (See https://www.jstor.org/stable/3618302) that every strongly prime number must be a permutation of a number of form $aa\cdots aab$, i.e. it cannot have more than two distinct digits.

4. The first few terms of a_n and b_n are $a_1 = 0, a_2 = 1, a_3 = 2, a_4 = 9$, and $b_1 = 1, b_2 = 1, b_3 = 3$. We notice that $a_2/b_1 = 1$, $a_3/b_2 = 2$, and $a_4/b_3 = 3$. We suspect the answer might be n.

   First, we find a recursion for a_n. Suppose we have a valid permutation of $1, 2, \ldots, n+1$, where $n+1$ is placed in position $j \neq n+1$. If j is placed in position $n+1$, then we get a valid permutation for the numbers $1, 2, \ldots, j-1, j+1, \ldots, n$. Since there are n choices for j this gives us na_{n-1} permutations. If $n+1$ is placed in the j-th position but j is not placed in position $(n+1)$, then thinking about position $n+1$ as position j, the condition becomes that j is not in the j-th position, which gives us a valid permutation for $1, 2, \ldots, n$. This gives us na_n valid permutations. Therefore, $a_{n+1} = na_n + na_{n-1}$ for all $n \geq 2$.

For b_n, note that a valid permuation for $1, 2, \ldots, n+1$ can be obtained from a permutation of $1, 2, \ldots, n$ in two ways.

- By inserting $n+1$ anywhere, except after n, in a valid permutation of $1, 2, \ldots, n$. This gives us nb_n permutations.
- By starting from a permutation of $1, 2, \ldots, n$ where there is precisely one i for which $i(i+1)$ appears and then inserting $n+1$ between i and $i+1$. Fixing i we observe that $i(i+1)$ is not allowed to appear before $i+2$ or after $i-1$, which means there are precisely b_{n-1} such permutations of $1, 2, \ldots, n$. Allowing $i = 1, 2, \ldots, n-1$, we obtain $(n-1)b_{n-1}$ valid permutations. Therefore, $b_{n+1} = nb_n + (n-1)b_{n-1}$.

Setting $c_n = (n-1)b_{n-1}$, we have $b_n = c_n + c_{n-1}$, which implies $c_{n+1} = nc_n + nc_{n-1}$, which is the same recursion as the one we obtained for a_n. Also, note that $c_2 = a_2 = 1$, and $c_3 = a_3 = 2$. Therefore, $c_n = a_n$, which proves $a_{n+1} = nb_n$. Thus, for all $n \geq 1$, we have $\frac{a_{n+1}}{b_n} = n$, and in particular $\frac{a_{2020}}{b_{2019}} = 2019$.

5. **First solution.** For every k, we have

$$\frac{a_k}{1 + a_{k+1} - a_{k-1}} = \frac{a_k(1 - a_{k+1} + a_{k-1})}{1 - (a_{k+1} - a_{k-1})^2} \geq a_k(1 - a_{k+1} + a_{k-1})$$

Summing this up over k and noticing that $\sum_{k=1}^{n} a_k(1 - a_{k+1} + a_{k-1}) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} (a_k a_{k-1} - a_k a_{k+1}) = 1 + 0 = 1$, we obtain the inequality.

Second solution. Using Cauchy-Shwarz Inequality we have

$$\left(\sum_{k=1}^{n} \frac{a_k}{1 + a_{k+1} - a_{k-1}}\right)\left(\sum_{k=1}^{n} a_k(1 + a_{k+1} - a_{k-1})\right) \geq \left(\sum_{k=1}^{n} a_k\right)^2 = 1$$

Note that $\sum_{k=1}^{n} a_k(1 + a_{k+1} - a_{k-1}) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} (a_k a_{k+1} - a_k a_{k-1}) = 1 + 0 = 1$. Thus, we obtain the desired inequality.