29th ANNUAL UNIVERSITY OF MARYLAND HIGH SCHOOL MATHEMATICS COMPETITION PART II SOLUTIONS

1. **Solution:** One way: Suppose the first three hobbits have a, b, and c rubles. Then $a+b+c=2007$. If the fourth hobbit has d rubles, then $b+c+d=2007$, which implies that $a=d$. If the fifth hobbit has e rubles, then $c+d+e=2007$. This implies that $e=b$. Similarly, the sixth has c, and the amounts continue to cycle through a, b, c. Therefore, the 98th has b, the 99th has c, and the 100th has a. Together, the 99th, 100th, and 1st have $2007=c+a+a$, so $a=b$. Also, the 100th plus the 1st plus the 2nd have $2007=a+a+b$. so $a=c$. Therefore, $a=b=c$, and they all have 669 rubles.

   Another solution: Let T be the total number of rubles among all 100 hobbits. Consider one hobbit, who has x rubles. The remaining 99 hobbits can be grouped into 33 groups of three, so these 99 total 33×2007 rubles. Therefore, $T = x+33\times 2007$. This implies that x is independent of the choice of hobbit, so all hobbits have the same amount, which must be 669.

2. **Solution:** One way: Let a be her age. Write $a=2^x3^y5^z\cdots$. Then $b^3 = (2/3)a^2 = 2^{2x+1}3^{2y-1}5^{3z}\cdots$. For this to be a cube, all the exponents must be multiples of 3. The smallest possibility is therefore $x=1, y=2, z=0, \ldots$. This yields $a=18$.

   The second smallest possibility for a is obtained by increasing one of the exponents. To ensure that the exponents in b^3 are multiples of 3, the exponent must be increased by 3. The smallest increase is obtained by increasing x to 4, which yields $a=2^4 3^2=144$.

   Another solution: Write $a=b\sqrt{3b/2}$. We see that $3b/2$ must be a square. The smallest b that works is $b=6$, which yields $a=18$. The second smallest is $b=24$, which yields $a=144$.

3. **Solution:** One way: Observe that the denominator factors as $n(n+1)(n+2)$. We claim that

$$\frac{1}{4}-\sum_{n=1}^{N}\frac{1}{n(n+1)(n+2)}=\frac{1}{2(N+1)(N+2)}$$

   for all $N\geq 1$. Since

$$\frac{1}{2(N+1)(N+2)}-\frac{1}{(N+1)(N+2)(N+3)}=\frac{1}{2(N+2)(N+3)},$$

   the claim is easily proved by induction. Letting $N=2007$ yields the desired inequality.

   Another solution: Write

$$\frac{1}{n(n+1)(n+2)}=\frac{1}{2}\left(\frac{1}{n}-\frac{2}{n+1}+\frac{1}{n+2}\right)$$

(this is a partial fraction decomposition). The sum for $n = 1$ to $n = 2007$ becomes

$$\frac{1}{2}\left(\left(1-\frac{2}{2}+\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right)+\cdots+\left(\frac{1}{2007}-\frac{2}{2008}+\frac{1}{2009}\right)\right).$$

The terms in the middle cancel, leaving

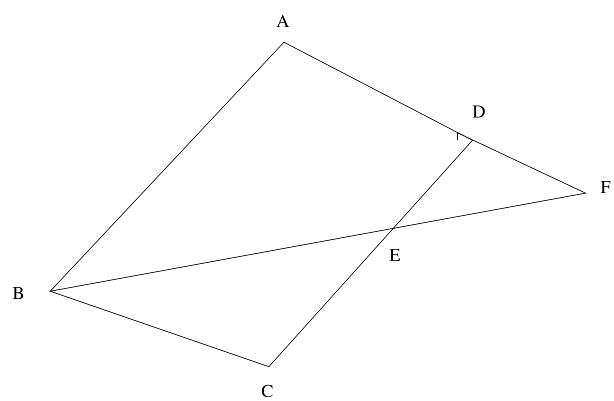
$$\frac{1}{2}\left(1-\frac{1}{2}-\frac{1}{2008}+\frac{1}{2009}\right)=\frac{1}{4}-\frac{1}{2\cdot 2009\cdot 2009}<\frac{1}{4}.$$

4. **Solution:** (a) Let P be any point on BC. Let x be the distance from P to AB and y be the distance from P to AC. Then $(AB)x + (AC)y = 2\times$ area of ABC. If $AB = BC$, then $x+y = 2(area)/(AB)$, which is independent of P. Conversely, suppose there are distinct points P_1 and P_2 such that $x_1 + y_1 = x_2 + y_2$. Then we have the system of equations

$$\begin{aligned}(AB)x_1 + (AC)y_1 &= (AB)x_2 + (AC)y_2\\ x_1 + y_1 &= x_2 + y_2.\end{aligned}$$

Multiply the second equation by AB and subtract to obtain $(AC - AB)y_1 = (AC-AB)y_2$. Since BC is not parallel to AB, we must have $y_1 \neq y_2$. Therefore, $AC = AB$, so the triangle is isosceles.

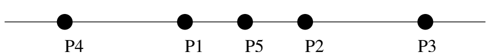
(b) Label the vertices of the quadrilateral $ABCD$.



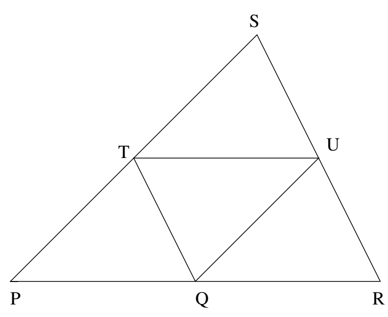
We'll first show that angle B equals angle D. Suppose that $BC \leq DC$ (the case $DC \leq BC$ is similar). Choose E on side CD so that $BC = BE$. Let F be the intersection of the extension of BE with the extension of AD. Let P be any point on BF. The solution of part (a) shows that the sum of the distances from P to sides CB and CD is independent of P. Since the sum of the distances to all four sides of the quadrilateral is constant, the sum of the distances from P to sides AB and AD is independent of P. By part (a), triangle ABF is isosceles. If $F = D$, then both ABD and CBE are isosceles, which implies that angle B equals angle D. Now suppose $D \neq F$. Then $\angle ADC = \angle AFE + \angle ABF$ (each side plus $\angle FDE$ is 180°). Also, $\angle EBC = \angle BEC = \angle DEF$, and $\angle AFE = \angle ABF$. Therefore, $\angle ADE = \angle ABF + \angle EBC = \angle ABC$, which says that angle D equals angle B in the quadrilateral.

Similarly, $\angle A = \angle C$. Since the sum of the four angles is 360°, we must have $2(\angle A + \angle B) = 360$, hence $\angle A + \angle B = 180$. This implies that DA is parallel to BC. Similarly, AB is parallel to CD. Therefore, $ABCD$ is a parallelogram.

5. **Solution:** Draw a line in the plane. We claim that there are three equally spaced points on this line of the same color: Choose two points P_1, P_2 on the line of the same color, say Red. Let $P_3 \neq P_1$ be the point on the line such that $P_1P_2 = P_2P_3$. If P_3 is Red, we are done. Otherwise, P_3 is Green. Now consider the point $P_4 \neq P_2$ such that $P_4P_1 = P_1P_2$. If P_4 is Red, we are done. Therefore, suppose P_4 is Green. Now consider the midpoint P_5 of P_1P_2. If P_5 is Red, then P_1, P_5, P_2 are equally spaced Red points. If P_5 is Green, then P_4, P_4, P_3 are equally spaced Green points. This proves the claim.



Let P, Q, R be the equally spaced points on the line, as in the claim.



Let's suppose they are all Red. Choose S so that PSR is similar to ABC. If S is Red, we are done. Therefore, assume that S is Green. Let T be the midpoint of PS and let U be the midpoint of RS. Then PTQ and QUR are similar to ABC. If either T or U is Red, we are done. Therefore, assume that T and U are Green. Then TSU is an all Green triangle similar to ABC.