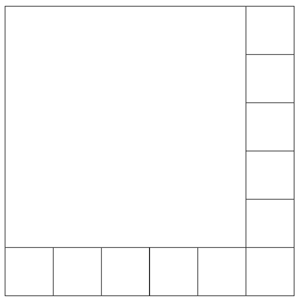
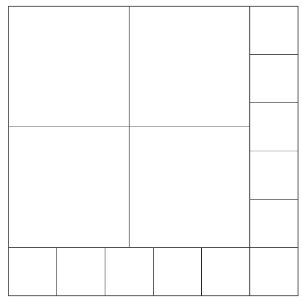
30th ANNUAL UNIVERSITY OF MARYLAND
HIGH SCHOOL MATHEMATICS COMPETITION
PART II
SOLUTIONS

1. **Solution:** One way: Divide the square into $2k+1$ small squares and one larger square, as in the left figure. This gives $2k+2$ squares, for any $k \geq 1$, so every even $n \geq 4$ is possible. Now divide the larger square into four smaller squares, as in the right figure. This gives $2k+5$ squares, for every $k \geq 1$, so every odd $n \geq 7$ is possible. Therefore, every $n \geq 6$ is possible.





   Another solution: If we can divide into n squares, we can subdivide one of the squares into 4 smaller squares and obtain $n+3$ squares. Since it is easy to divide the original square into 4 squares, we obtain $7, 10, 13, \ldots$ this way. Similarly, we can explicitly divide the original square into 6 squares (the case $k = 1$ in the first solution) and obtain $9, 12, 15, \ldots$. We can also explicitly divide the original square into 8 squares (the case $k = 2$ in the first solution) and obtain $8, 11, 14, \ldots$. Therefore, all $n \geq 6$ are possible.

2. **Solution:** One way: Note that $5^2 = 25$, $35^2 = 1225$, and $335^2 = 112225$. Therefore, we guess that $x = 333\cdots3335$ satisfies $x^2 = n$, where there are 2008 threes. Multiply $x \times x$ by the standard multiplication algorithm:

						3	3	3	···	3	3	3	5
					×	3	3	3	···	3	3	3	5
					1	6		···	6	6	6	7	5
				1	0	0	0	0	···	0	0	5	
			1	0	0	0		···	0	0	5		
		1	0	0	0		···	0	0	5			
	1	0	0	0		···	0	0	5				
			···			···		···					
			···			···							
1	0	0	0	···	0	5							
1	1	1	1	···1	2	2	2	···	2	2	2	2	5

   An easy count shows that there is the correct number of ones and twos, so $x^2 = n$.

Another solution: In the notation of the first solution, we have $3x = 1000\cdots0005$, so $9x^2 = 1000\cdots10\cdots25$. Divide by 9 to get n, so $n = x^2$.

A third solution: $111\cdots111$ (with 2008 ones) equals $(10^{2008}-1)/9$. Therefore,

$$n = 10^{2010}((10^{2008}-1)/9) + 200((10^{2008}-1)/9) + 25.$$

Therefore,

$$9n = 10^{4018} + 10^{2010} + 25 = (10^{2009}+5)^2.$$

Therefore, n is a perfect square.

3. **Solution:** Note that for any initial n, either Player I or Player II has a winning strategy: For example, if Player II does not have a winning strategy, then there is a sequence of moves by Player I for which Player II cannot respond to obtain a win. On the other hand, if Player I does not have a winning strategy, then Player II has a winning response to every sequence of moves.

   Suppose there are only finitely many n for which Player II has a winning strategy (there exist such n, for example, $n = 2$). Let N be the largest such n. Let $n_1 = N + 1 + N^2$. Since $n_1 < (N+1)^2 = 2N + 1 + N^2$, Player I can remove at most N^2 stones. Therefore, there will be at least $n_1 - N^2 = N + 1 > N$ stones remaining. By the choice of N, the first player has a winning strategy for this number of stones. Since it is Player II's turn to play, Player II is effectively the first player and therefore can use the winning strategy to win the game. Since $n_1 > N$, we see that N is not the largest n for which Player II has a winning strategy. This contradiction shows that the set of winning n's for Player II must be infinite.

4. **Solution:** (a) We need two facts. First, if the midpoints of the sides of a convex quadrilateral are connected, the result is a parallelogram. (This is proved by noting that the line joining the midpoints of two sides of a triangle is parallel to the third side. In the present situation, the "third side" is a diagonal of the quadrilateral.) Second, the diagonals of a parallelogram bisect each other. This is well known.

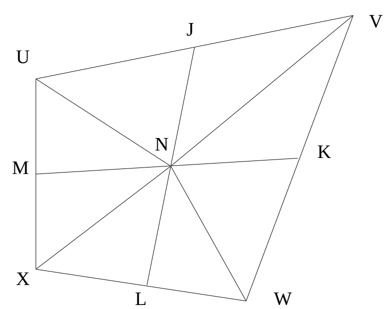
   Let the dividing points on AD be $R_1, R_2, \ldots, R_7$, and let the dividing points on BC be $S_1, S_2, \ldots, S_7$.

   First we show that P_4Q_4 is divided into 8 equal segments. Since $P_4S_4Q_4R_4$ is a parallelogram, its diagonals P_4Q_4 and R_4S_4 bisect each other. Call the point of intersection O. Then P_4O and OQ_4 have equal lengths. Now consider $R_2P_4S_2O$, which connects the midpoints of a quadrilateral, and hence is a parallelogram. Its diagonals are R_2S_2 and P_4O, which therefore bisect each other, say at T.

Considering the parallelogram $R_1P_4S_1T$ shows similarly that R_1S_1 bisects P_4T. The fact that the other lines R_iS_i divide P_4Q_4 into equal length segments follows in the same way. Similarly, the lines P_iQ_i divide R_4S_4 into equal segments.

Now apply the same argument to each of the quadrilaterals AP_4OR_4, P_4BS_4O, S_4CQ_4O, and DR_4OQ_4 to show that each of the lines P_2Q_2, P_6Q_6, R_2S_2, and R_6S_6 is divided into 8 equal segments. Finally, apply the argument to each of the 16 two-by-two quadrilaterals to find that each of the lines is divided into 8 equal segments, as desired.

(b) Suppose a convex quadrilateral $UVWX$ is divided into a two-by-two checkerboard by dividing each side into two equal segments. Let points J, K, L, M be as in the diagram. Let N be the intersection of JL and LM. The triangles UNJ and VNJ have opposite colors but the same areas. Similarly for the pair VNK and WNK, the pair WNL and XNL, and the pair XNM and UNM. Therefore, in the two-by-two case, the total black area equals the total white area.



The eight-by-eight is divided into 16 two-by-two checkerboards, each of which has each of its sides divided into two equal segments, by part (a). Since the two-by-two case is true for each of these smaller quadrilaterals, the total white area equals the total black area.

5. **Solution:** If 10^h has 2008 digits in base 2, and 10^k has 2008 digits in base 5, then

$$2^{2007} < 10^h < 2^{2008}$$
$$5^{2007} < 10^k < 5^{2008}$$

Multiplying yields $10^{2007} < 10^{h+k} < 10^{2008}$, which is impossible since 10^{2007} and 10^{2008} are successive powers of 10, so there cannot be another power of 10 between them. Therefore, we cannot have 2008 digits both in base 2 and in base 5. Now suppose no power of 10 has 2008 digits in base 2 or in base 5. Then

$$10^h < 2^{2007} < 2^{2008} < 10^{h+1}$$
$$10^k < 5^{2007} < 5^{2008} < 10^{k+1}$$

for some integers h, k. Multiplying yields $10^{h+k} < 10^{2007} < 10^{2008} < 10^{h+k+2}$, which is impossible since there cannot be two distinct powers of 10 between 10^{h+k} and 10^{h+k+2}.

In case you're wondering, 10^{1403} has 2008 digits in base 5.