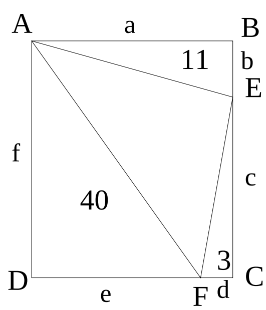
31st ANNUAL UNIVERSITY OF MARYLAND
HIGH SCHOOL MATHEMATICS COMPETITION
PART II
SOLUTIONS

1. **Solution:** (a) Either at least two of the integers are odd or at least two of the integers are even. The average of two odd integers is an integer, and the average of two even integers is an integer, so in both cases we obtain an integer as an average.
(b) When an integer is divided by 3, the remainder is 0, 1, or 2. Among the five integers, either three give the same remainder when divided by 3, or all three possible remainders occur. If all three have the same remainder, their sum is a multiple of 3 plus 3 times this common remainder, so the sum is a multiple of 3. To get the average, divide by 3, which gives an integer average. If all three remainders occur, the sum of these three numbers is a multiple of 3 plus the sum of the three remainders: $0+1+2=3$. Therefore, the sum is a multiple of 3, so the average is an integer.

2. **Solution:** $f(7)+f(11)=7^2+7a+b+11^2+11a+b=7^2+11^2+18a+2b$. Similarly, $g(7)+g(11)=7^2+11^2+18c+2d$. Since $f(7)+f(11)=g(7)+g(11)$, we have $18a+2b=18c+2d$, which implies that $9a+b=9c+d$.
(a) $f(9)=9^2+9a+b=9^2+9c+d=g(9)$.
(b) Suppose $f(z)=g(z)$ for some $z\neq 9$. This means that $z^2+az+b=z^2+cz+d$, so $az+b=cz+d$. Subtracting the equation $9a+b=9c+d$ yields $a(z-9)=c(z-9)$. Since $z\neq 9$, we obtain $a=c$. The equation $9a+b=9c+d$ now implies that $b=d$, so $f(x)$ and $g(x)$ are the same polynomial, contradicting the assumption that there are different. Therefore, there is no $z\neq 9$.

3. **Solution:** Label the pieces of the sides as in the figure. Then $ab=22$, $cd=6$,



$ef=80$, $a=d+e$, $f=b+c$. Therefore,

$$a=d+e=\frac{6}{c}+\frac{80}{f},$$

so $c=6f/(af-80)$. Therefore,

$$f=b+c=\frac{22}{a}+\frac{6f}{af-80},$$

which yields $af = 22 + 6af/(af - 80)$, hence $0 = (af)^2 - 108(af) + 1760 = (af - 20)(af - 88)$. Since $af = 20$ gives too small an area, we find that the area is $af = 88$.

4. **Solution:** There are 2^{49} ways to put the markers in the first 7 rows and columns. Each such arrangement can be completed to a solution as follows: Put markers in the last row so that each of the first 7 columns has an odd number of markers. Leave the last square blank. Now put markers in the last column so that each of the 8 rows has odd number of markers. At this point, every row and the first 7 columns have odd numbers of markers. Is the last column odd? Adding up using the rows, we see that the total number of markers is the sum of 8 odd numbers, hence even. Using the columns instead, we have the sum of 7 odd numbers plus the number of markers in the last column. Since this total is even, the number for the last column is odd. The choices for the 49 squares were arbitrary, and the remaining placements of the markers were forced. Therefore, there are 2^{49} ways to place the markers.

5. **Solution:** (a) Person 1 guesses that his hat has the same color as the hat on Person 2. Person 2 guesses that his hat has a different color than the hat on Person 1. Exactly one of the two people will be correct.
(b) Form 1004 pairs. In each pair, use the strategy from (a). Then exactly 1004 people will be correct.
(c) Make a 2^{2009} by 2009 array, where the rows are indexed by all possible combinations of hats and the columns are indexed by the 2009 people. Put a 1 in each entry where the person (named by the column) gets the correct answer for the combination named by the row, and put 0's for the incorrect answers. For each of the 2^{2008} combinations that a person sees, there are two possibilities for his hat color. One corresponds to his guess and one doesn't. Therefore, each column has 2^{2008} entries that are 1 and 2^{2008} entries that are 0. This means that exactly 1/2 of the entries in the array are 1's. If at least 1005 people guess correctly for each combination, then each row has more than 1/2 of it entries equal to 1. This means that more than 1/2 of the entries of the array are 1's, which is impossible.