THE 26th ANNUAL (2004) UNIVERSITY OF MARYLAND

HIGH SCHOOL MATHEMATICS COMPETITION

ANSWERS

1. The numbers are $3^2 = 9$, $2^1 = 2$, $3^1 = 3$, $1^8 = 1$, and $2^4 = 16$. The answer is **(e)**.

2. Add the two equations together to obtain $13x + 13y = 6006$. Divide by 13 to get $x + y = 462$. The answer is **(c)**.

3. S has area s^2 and perimeter $4s$, and C has area πr^2 and perimeter $2\pi r$. By assumption, $4s = 2\pi r$, so $s = \pi r/2$. This means the area of S is $s^2 = \pi^2 r^2/4$. The ratio A_S/A_C equals $(\pi^2 r^2/4)/(\pi r^2) = \pi/4$. The answer is **(e)**.

4. There are 3 possibilities for the first digit (1, 4, 9) and 4 possibilities for the second digit (0, 1, 4, 9). Therefore, there are $3 \times 4 = 12$ numbers. The answer is **(d)**.

5. We are told that 5 geese produce 55 eggs in 5555 days. In this time period, each of the 5 geese produces an average of 11 eggs. In other words, one goose produces 11 eggs in 5555 days. Dividing by 11 shows that one goose produces 1 egg in 505 days. The answer is **(c)**.

6. The customer obtains $120 - 44 = 76$ inches of chain for 600 cents. Therefore, he pays 600/76 cents per inch. Multiply by 12 to obtain $12 \times 600/76 = 94.7$ cents per foot. The answer is **(d)**.

7. Note that $|2-x|$ is the distance from 2 to x, and $|x-y|$ is the distance from x to y, and $|y-2004|$ is the distance from y to 2004. Therefore, the sum is the distance traveled when going from 2 to x to y to 2004. The shortest this can be is the distance from 2 to 2004, which is 2002. The answer is **(c)**.

8. Form a 3-4-5 right triangle, with 3 opposite the angle x. Then $\sin x = 3/5$ and $\tan x =$ opposite/adjacent $= 3/4 = 0.75$, so the answer is **(c)**.

9. The 100 pounds of fresh cherries contain 99 pounds water and 1 pound of other material. When the 1 pound of other material is 2%, the water must be 49 pounds, so the total weight is 50 pounds. The answer is **(b)**.

10. If the polynomial has two integer roots, then it factors as $(X + r)(X + s)$ for two integers r, s. This means that $rs = 30$ and $r + s = b$. Since $b > 0$, we must have $r, s > 0$. The possible pairs (r, s) are $(1, 30), (2, 15), (3, 10), (5, 6), (6, 5), (10, 3), 15, 2), (30, 1)$. These give the four values 31, 17, 13, 11 for b. The answer is **(d)**.

11. Since $5 < 6$, we have $\log_5 6 > 1$ and $\log_6 5 < 1$. Therefore answer (a) is larger than answer (b), and answer (d) is larger than answers (c) and (e). But clearly, (d) is larger than (a). The answer is **(d)**.

12. For a quick estimate, replace $\sqrt{26}$ by 5, and $\sqrt{65}$ by 8. Also, replace $\sqrt[3]{26}$ by 3 and $\sqrt[3]{65}$ by 4. Then a is approximately $\sqrt{5+4} = 3$, b is approximately $\sqrt{3+8} = \sqrt{11} > 3$, and c is approximately $\sqrt[3]{5+8} < 3$. This yields $c < a < b$. To prove $c < a$ rigorously, note that $c < \sqrt[3]{6+9} < 3$ and $a > \sqrt{5+4} = 3$. To prove $a < b$, it suffices to prove that $a^2 < b^2$. One way is the following: Note that the polynomial $X^3 - X^2 = X^2(X-1)$ gets larger as $X > 1$ increases, since both X^2 and $X - 1$ are increasing functions. Since $26^{1/6} < 65^{1/6}$, we find that $(26^{1/6})^3 - (26^{1/6})^2 = \sqrt{26} - \sqrt[3]{26}$ is smaller than $(65^{1/6})^3 - (65^{1/6})^2 = \sqrt{65} - \sqrt[3]{65}$. Rearranging yields $a^2 < b^2$. The answer is **(e)**.

13. If A is true and B, C are false, then both (c) and (e) are true. If B is true and A, C are false, then (a) and (b) are true. If C is true and A, B are false, then only (d) is true. The answer is **(d)**.

14. The original fee was 15. Let n be the original number of customers. The original money collected was $15n$. After the fee was reduced to f, the money collected was $(f)(1.5n)$. By assumption, $(f)(1.5n) = 1.25(15n)$. Divide by $1.5n \neq 0$ to obtain $f = 12.5$. The answer is **(b)**.

15. The total loss is $100 + 200 + 300 + \cdots + 10000 = 100(1 + 2 + 3 + \cdots 100)$. Divide by 100 to obtain the average loss, which is $1 + 2 + \cdots 100 = 5050$ (use, for example, the formula $1 + 2 + \cdots + n = n(n+1)/2$; alternatively, pair 1 with 99, 2 with 98, etc., and then add). The answer is **(a)**.

16. Since $X^2 - 9X + 3 = (X - r)(X - s) = X^2 - (r + s)X + rs$, we have $r + s = 9$ and $rs = 3$. Therefore, $r^2 + s^2 = (r + s)^2 - 2rs = 9^2 - 6 = 75$ and $r^2s^2 = 3^2 = 9$. Therefore, $X^2 + bX + c = (X - r^2)(X - s^2) = X^2 - (r^2 + s^2)X + r^2s^2 = X^2 - 75X + 9$. The answer is **(e)**.

17. The last digits of $7, 7^2, 7^3, 7^4, \ldots$ cycle through the four numbers 7, 9, 3, 1. Therefore, the last digit of 7^n depends only on the remainder when the exponent n is divided by 4. When $7, 7^2, 7^3, \ldots$ are divided by 4, the remainders alternate between 3 and 1. Therefore, 7^7 is 3 more than a multiple of 4. It follows that the last digit of $7^{(7^7)}$ is the third entry in 7, 9, 3, 1, namely 3. The answer is **(b)**.

18. One sixteenth of the octagon is a right triangle with hypotenuse 1 (the radius of the circle) and with one angle $360/16 = 22.5^\circ$. The legs are therefore $\sin 22.5$ and $\cos 22.5$. The area of the triangle is $(1/2)\sin 22.5 \cos 22.5$. Since $2 \sin x \cos x = \sin 2x$, the area of the triangle is $(1/4)\sin 45 = \sqrt{2}/8$. The octagon has area $16(\sqrt{2}/8) = 2\sqrt{2}$. The answer is **(e)**.

19. Let d be the distance from A to B, let the first hiker's speed be a, and let the second hiker's speed be b. Let t be the time when the second hiker arrives at A. When the two hikers meet at 3pm, the first hiker has walked a distance $3a$ and the second hiker has walked $3b$. Since they were coming from A and B, the distances they walked add up to d, so $3a + 3b = d$. At time $t - 2.5$, the first hiker has walked a distance $(t - 2.5)a$, which is also the entire distance d, so $(t - 2.5)a = d$. Similarly, $tb = d$. Therefore, $a = d/(t - 2.5)$ and $b = d/t$. Substitute these into $3a + 3b = d$ and divide n=by d to obtain $(1/(t - 2.5) + 1/t = 1$. This yields the quadratic equation $t^2 - 8.5t + 7.5$. The roots are $t = 1$ and $t = 7.5$. Since $t = 1$ is too early, we must have $t = 7.5$. The answer is **(d)**.

20. The first jump must be to (1,1) and the sixth jump is from (5,1). First suppose the second jump is down to (2,0). Then the third jump must be to (3, 1). There are two ways to do the fourth jump, and then the fifth jump is completely determined. This yields two paths. Second, suppose the second jump is up to (2,2). If the third jump is up to (3,3), then the next two jumps are down This yields one path. If the third jump is down to (3,1), then there are two ways to do the fourth jump and the fifth jump is determined. This yields two paths. The overall total is 5 paths. The answer is **(a)**.

21. Suppose that removing the last two digits of k^2 yields the square s^2. Then $(10s)^2$ has the same digits as k^2 except that the last two digits are 0's. This means that $0 < k^2 - (10s)^2 < 100$. But $k^2 - (10s)^2 = (k + 10s)(k - 10s) \geq k + 10s \geq 10s + 10s = 20s$, so $20s < 100$. Therefore, $s \leq 4$, and $k^2 < 100 + (10s)^2 \leq 1700$. This implies that $k \leq 41$. In fact, $41^2 = 1681$, so $k = 41$ works. The answer is **(b)**.

22. The graph of $x^2 = y^2$ is two lines in the plane. The graph of $(x-A)^2 + y^2 = 1$ is a circle of radius 1 centered at $(A, 0)$. If there are exactly 3 solutions, then the circle intersects the lines at three points. If (x, y) is an intersection point, then so is $(x, -y)$, which means the intersections occur in pairs. The only way to get an odd number of intersections is to have one of the points be $(0, 0)$. The center of the circle is a distance 1 away, so $A = \pm 1$. The answer is **(c)**.

23. Each seven-digit number is congruent mod 9 to the sum of its digits, which is $1+2+\cdots+7 = 28$. Since 28 is congruent to 1 mod 9, each number is congruent to 1 mod 9. If a and b are two such numbers and $a = kb$, then a is congruent to 1 and kb is congruent to k mod 9. Therefore, k is congruent to 1 mod 9. Since $k = 1$ is not allowed and since clearly $k < 10$, this is impossible. Therefore, there are no pairs a, b. The answer is **(a)**.

24. Statement (i) is true: Let the first triangle be equilateral with sides 1/2 inch. Let the vertices of the second triangle be $(0, 0), (10000, .0000001), (20000, 0)$ (units are inches). Then the area of the second triangle is .001, which is less than the area of the first triangle. Statement (ii) is true: Consider the triangle with vertices at $(0, 0), (.1, 10000), (0, 20000)$. The area is 1000. Each side has length more than 10000. Since the altitude times the appropriate side is twice the area, the altitudes all must be less than .2. Statement (iii) is false: If each altitude is longer than 2 inches and the area is less than 1, then each side is less than 1. Since the altitude is the shortest distance from a vertex to the line of the opposite side, the altitude must be at most the length of any side coming out of the vertex. Therefore, the altitudes cannot have length more than 1, which is a contradiction. The answer is **(c)**.

25. Let d be the greatest common divisor. Then d is a divisor of $m - 10^{2004-666}n - 10^{2004-2\cdot 666}n - 10^6 n = 111111$. Since 111111 is a divisor of m and n, we have $d = 111111$. The answer is **(c)**.