

THE 27th ANNUAL (2005) UNIVERSITY OF MARYLAND
HIGH SCHOOL MATHEMATICS COMPETITION
PART I SOLUTIONS

1. Suppose there are d dogs and p people. Then $4d + 2p = 82$ and $d + p = 26$. Doubling the second equation and subtracting from the first yields $2d = 30$, or $d = 15$. The answer is **d**.
2. The set T is the inside of a square with corners at $(0, 4)$, $(0, -4)$, $(4, 0)$, $(-4, 0)$. Each side has length $\sqrt{4^2 + 4^2} = \sqrt{32}$. The area is $(\sqrt{32})^2 = 32$. The answer is **e**.
3. Each of the last four numbers is > 1 . Since $3^2 > 2^3$, taking 6th roots yields $3^{1/3} > 2^{1/2}$. Similarly, $3^4 > 4^3$, so taking 12th roots yields $3^{1/3} > 4^{1/4}$. Finally, $3^5 > 5^3$, and taking 15th roots yields $3^{1/3} > 5^{1/5}$. The answer is **c**.
4. Dividing by 2 shows that 50 wolves can devour 25 sheep in 2 weeks. Multiply by 25 to find that 50 wolves can devour $25 \times 25 = 625$ sheep in 50 weeks. Another way: it takes 4 wolf-weeks to devour a sheep. Therefore 2500 wolf-weeks is enough to devour $2500/4 = 625$ sheep. The answer is **d**.
5. In each pair of months, the weight is multiplied by $3/2$ and then by $1/2$, so the result is $3/4$. This is done for six pairs of months, so the weight is $65536 \times (3/4)^6$. Since $65536 = 2^{16}$ and $4^6 = 2^{12}$, this simplifies to $3^6 \times 2^4 = 729 \times 16 = 11664$. The answer is **a**.
6. When the linear dimensions are increased by a factor of 20, the surface area is increased by a factor of $20^2 = 400$. The answer is **d**.
7. Let x, y, z be the width, length, and height, respectively. Then $xy = 120$, $xz = 96$, and $yz = 80$. Therefore $z/x = 80/120 = 2/3$ and $z^2 = (xz)(z/x) = (96)(2/3) = 64$. Therefore, $z = 8$. The answer is **a**.
8. Take the 20th root of each number to obtain the numbers $10^5 = 100000$, $20^4 = 160000$, $30^3 = 27000$, $40^2 = 1600$, 50. The second is largest, so the answer is **b**.

9. Since $1 + \tan^2 \theta = \sec^2 \theta = 1/\cos^2 \theta$, the expression equals $\sin^2 \theta + \cos^2 \theta = 1$, independent of the value of $\sin \theta$. The answer is **e**.
10. Expand the sum to obtain $\log 2 - \log 1 + \log 3 - \log 2 + \log 4 - \log 3 + \cdots + \log 100 - \log 100$. All the terms cancel, except for $\log 100 - \log 1 = 2 - 0 = 2$. The answer is **b**.
11. Suppose there are usually s students in the math class. Since exactly $1/4$ know how to play poker, s must be a multiple of 4. On Wednesday, $s - 7$ students are present and exactly $1/5$ know how to play poker, so $s - 7$ is a multiple of 5. The only multiple of 4 between 15 and 40 that is 7 more than a multiple of 5 is $s = 32$. On Wednesday, there were $s - 7 = 25$ students, and 5 knew how to play poker. The answer is **c**.
12. Let x be the number of two-looped tails and y be the number of three-looped tails. Then $2x + 3y = 100$. Therefore, y must be even, so (e) is correct. Therefore, (a) is wrong. The example, $x = 5, y = 30$ shows that (b), (c) can be false. The example $x = 2, y = 32$ shows that (d) can be false. The answer is **e**.
13. Twice the area is the altitude times the base. The longest base therefore has the shortest altitude. Use the two legs to calculate twice the area to be $3 \times 4 = 12$. Use the longest side, namely the hypotenuse of length 5, to calculate twice the area. The altitude must be $12/5$. The answer is **d**.
14. Solve $x^2 = bx - b$, so $x^2 - bx + b = 0$. This has exactly one root when the discriminant is 0, so when $0 = (-b)^2 - 4b = b^2 - 4b = b(b - 4)$. Since $b \neq 0$ by assumption, we have $b = 4$. The answer is **d**.
15. Let AB have length s and let ZY have length t . Angle ZAY has 120 degrees. By the law of cosines, $t^2 = (s)^2 + (2s)^2 - 2(s)(2s) \cos 120 = 7s^2$. Since XYZ and ABC are equilateral, the ratio of their areas is the square of the ratios of their sides, which is $t^2/s^2 = 7$. Therefore, XYZ has area 7. The answer is **c**.
16. Each rock is the endpoint of two arcs, so the total new mass added by each alien is twice the mass already present. Therefore, each alien triples the mass. At the start, the mass is 2 grams. After n aliens, the mass is 2×3^n grams. If $2 \times 3^n > 6 \times 10^{29}$, then $3^{n-1} > 10^{29}$. As a

rough estimate, using $3^2 \approx 10$, we get $n \approx 59$. This can be made more precise: $3^2 < 10$, so $3^{58} < 10^{29}$. Therefore, $n > 59$. However, $3^3 > 10$, so $3^{87} > 10^{29}$, which implies that $n \leq 88$. Therefore, $58 < n \leq 88$. The answer is **b**.

17. (a) and (c) are equivalent. Since exactly one statement is false, both of these must be true. Since (a) implies (e), we must have (e) true. If (b) is true, then (a) and (b) imply (d), so all statements are true. Therefore, (b) must be the false statement. An example where it is false is $A = 1, B = 7, C = 3, D = 2$. The answer is **b**.
18. Let the rectangle have length x and width y . Then $xy = 3$. The sum of the diagonals, by the Pythagorean theorem, is $2\sqrt{x^2 + y^2}$. Since $0 \leq (x - y)^2 = x^2 + y^2 - 2xy = x^2 + y^2 - 6$, we have $x^2 + y^2 \geq 6$, so the sum of the diagonals is at least $2\sqrt{6}$. This value actually occurs for the square of side $\sqrt{3}$, so the answer is **b**.
19. Let x_n be the expression. Then $x_{n+1} = x_n(\log n)/10$. Therefore, $x_{n+1} \leq x_n$ when $\log n \leq 10$. This happens for $n \leq 10^{10}$. When $\log n > 10$, we have $x_{n+1} > x_n$. Therefore, the smallest value occurs for $n = 10^{10}$ and for $n = 10^{10} - 1$. The answer is **b**.
20. Let t be the time required to finish the job, let a be the amount of time Paul spends painting and let e be the amount of time Peter spends painting. Then $20a + 40e = 4000$, and $30(t - a) + 15(t - e) = 1500$. The second equation yields $3t = 100 + 2a + e$. Substituting from the first equation yields $3t = 500 - 3e$. Since $e \leq t$, we have $3t \geq 500 - 3t$, so $t \geq 250/3$. If we let $e = 250/3$ and $a = 100/3$, then Paul and Peter finish the job in $250/3$ hours. The answer is **c**.
21. Fix one member, M . If one of the other 8 members is on a committee with M , then this is the only committee shared with M , by the hypothesis. Therefore, the remaining 8 members who are on committees with M are in disjoint 2-element subsets. There are at most 4 such subsets, so M can be on at most 4 committees. This yields an upper bound of 9×4 committee memberships. Divide by 3, the number of members of each committee, to obtain 12, the upper bound for the number of committees. It is possible to obtain 12: put the 9 members into a 3×3

array:

$$\begin{array}{ccc} A & B & C \\ D & E & F \\ G & H & I \end{array} .$$

Each row, each column, and each generalized diagonal gives a committee. For example, ADG , BDI , and BFG are three such committees. This yields 12 committees satisfying the hypotheses. The answer is **c**.

22. If three people have not shaken hands with each other, but everyone else has shaken hands with everyone, then there are 2002 people who have shaken hands with everyone, so the minimum is at most 2002. Suppose there are two people, A and B , who have not shaken hands with each other. There cannot be another such pair, C and D , disjoint from A and B , since then $ABCD$ is a set of 4 people and none has shaken hands with all of the other three. Therefore, all of the remaining 2003 people have shaken hands with each other. If each of these 2003 has shaken hands with both A and B , then there are 2003 people who have shaken hands with everyone. Suppose now that there is someone, $E \neq B$, who has not shaken hands with A . If F is anyone other than A, B, E , then in the group of $ABEF$, F must have shaken hands with the other three. Therefore, each of the 2002 possible people F has shaken hands with all 2005 people. Therefore, the minimum is at least 2002. The answer is **d**.
23. Fix an island I . There are at most 3 islands that are connected by bridges to I . Each of these three is connected to I and at most two other islands. This accounts for all the islands, since each island can be reached from I by at most two bridges. We therefore have at most 10 islands. It remains to show that this is possible. This can be done by trial and error, but here is another way: Put one smaller pentagon inside another larger one, with a $1/10$ turn. The vertices represent the 10 islands. Connect each outer vertex to one of the two closest inner vertices and also connect each outer vertex to the farthest inner vertex. This configuration satisfies the hypotheses. Therefore, the answer is **d**.
24. If $s > \sqrt{2}$, then $x > \sqrt{2}$ and $1/y > \sqrt{2}$, so $y + 1/x < \sqrt{2}$, hence $s < \sqrt{2}$. Contradiction. Therefore, $s \leq \sqrt{2}$. If $x = \sqrt{2}$ and $y = 1/\sqrt{2}$, then $y + 1/x = \sqrt{2}$, so $s = \sqrt{2}$. The answer is **a**.

25. If all judges gave 1st place to the same participant, then $c_1 = 9$. If exactly 2 people got 1st places, one of them got at least 5 ones. His score is at most $5 \times 1 + 4 \times 4$ (the worst he can get from other judges). So $c_1 \leq 5 + 16 = 21$. If 3 people got 1st places, the sum of their scores is at most $9 \times 1 + 9 \times 4 + 9 \times 3 = 72$. So $c_1 \leq 24$. If 4 people got 1st places, then the sum of their scores is at least $9 \times 1 + 9 \times 4 + 9 \times 3 + 9 \times 2 = 90$, and $c_1 \leq 22$. Five or more 1st places is not possible. Example for $c_1 = 24$: Each of the top 3 gets the scores 1,1,1,3,3,3,4,4,4 (not in the same order). The next gets 2,2,2,2,2,5,5,5,5 and the next gets 5,5,5,5,5,2,2,2,2. The rest get arbitrary grades with difference ≤ 3 . This gives $c_1 = 24$. The answer is **e**.