1. Archimedes, Euclid, Fermat, and Gauss had a mathematics competition. Archimedes said, "I did not finish 1st or 4th." Euclid said, "I did not finish 4th." Fermat said, "I finished 1st." Gauss said, "I finished 4th." There were no ties in the competition, and exactly three of the mathematicians told the truth. Who finished first and who finished 4th? Justify your answers.

   Solution: Call the mathematicians A, E, F, G. If A is lying, then he finished 1st or 4th, so F or G is also lying. Since only one person is lying, we see that A must be telling the truth. If E is lying, then he finished 4th, so G is also lying. Therefore, E is telling the truth. Since A and E are telling the truth, either F or G finished 4th. If F finished 4th, then F and G are lying, which is contrary to assumption. Therefore, G finished 4th, and he is telling the truth. So F is lying, and he did not finish 1st. Since A is telling the truth, he did not finish first. Therefore, E finished first. In summary, Euclid finished first and Gauss finished last.

2. Find the area of the set in the xy-plane defined by $x^2 - 2|x| + y^2 \leq 0$. Justify your answer.

   Solution: First consider $x \geq 0$. Adding 1 to each side of the equation yields $x^2 - 2x + 1 + y^2 \leq 1$, which is $(x-1)^2 + y^2 \leq 1$. This describes a disc of radius 1 centered at $(1, 0)$. All of the points in this disc have $x \geq 0$. Now consider $x \leq 0$. Then $|x| = -x$, so we have $x^2 + 2x + y^2 \leq 0$. This becomes $(x+1)^2 + y^2 \leq 1$, which is a disc centered at $(-1, 0)$, and all of the points in the disc have $x \leq 0$. The area of each disc is π, so the total area is 2π.

3. There is a collection of 2004 circular discs in the plane. The total area covered by the discs is 1 square meter. Show that there is a subcollection S of discs such that the discs in S are non-overlapping and the total area of the discs in S is at least 1/9 square meter.

   Solution: Let D_1 be the largest disc (if there is a tie, choose one of them arbitrarily). Expand the radius of D_1 by a factor of 3 to obtain an expanded disc E_1 and discard all discs that lie completely within E_1. Let D_2 be the largest disc that is not discarded. Expand its radius by a factor of 3 and discard all discs that lie completely within this expanded disc E_2. Now take the largest disc D_3 that is not discarded and continue the procedure until all discs have been discarded. The sum of the areas of E_1, E_2, ... must be at least 1 since we have discarded all discs. Since the area of an expanded disc is 9 times the area of the original disc, the sum $\text{Area}(D_1) + \text{Area}(D_2) + \cdots$ must be at least 1/9. Consider two discs D_i and D_j, with $i < j$. Suppose D_i and D_j overlap. Since the radius of D_i is at least as large as the radius of D_j, this means that D_j lies completely within the expanded circle E_i. But then D_j would have been discarded at the ith step, which is not the case. Therefore, the circles are non-overlapping.

Another way to look at this: We take the largest disc and throw out all discs that overlap with it. Then take the largest remaining disc and throw out all discs that overlap with it. Continue in this manner. The discs that remain have total area at least 1/9, by the reasoning given above.

4. Let S be the set of all 2004-digit integers (in base 10) all of whose digits lie in the set $\{1, 2, 3, 4\}$. (For example, $12341234 \cdots 1234$ is in S.) Let n_0 be the number of $s \in S$ such that s is a multiple of 3, let n_1 be the number of $s \in S$ such that s is one more than a multiple of 3, and let n_2 be the number of $s \in S$ such that s is two more than a multiple of 3. Determine which of n_0, n_1, n_2 is largest and which is smallest (and if there are any equalities). Justify your answers.

   Solution: Among the m-digit integers with digits lying in $\{1, 2, 3, 4\}$, let A_m be the set of integers where the sum of the digits is a multiple of 3, let B_m be the set of integers where the sum of the digits is one more than a multiple of 3, and let C_m be the set of integers where the sum of the digits is two more than a multiple of 3. Let a_m, b_m, c_m be the number of elements in A_m, B_m, C_m, respectively. In passing from m-digit numbers to $(m+1)$-digit numbers, we place a 1, 2, 3, or 4 at the beginning of an m-digit number. An element of A_{m+1} can arise from placing a 3 on an element of A_m, a 2 on an element of B_m, or a 1 or 4 on an element of C_m. Therefore,

$$a_{m+1} = a_m + b_m + 2c_m = (a_m + b_m + c_m) + c_m.$$

   Similarly,

$$b_{m+1} = (a_m + b_m + c_m) + a_m, \qquad c_{m+1} = (a_m + b_m + c_m) + b_m.$$

Note that $a_1 = 1, b_1 = 2, c_1 = 1$. Suppose that $a_m = b_m - 1 = c_m$. Then the recurrence relations imply that $a_{m+1} = b_{m+1} = c_{m+1} - 1$. Similarly if a_m or c_m is one more than the other two. So the largest element cycles through a, b, c, and the other two numbers are equal. Since 2004 is a multiple of 3, $a_{2004} - 1 = b_{2004} = c_{2004}$. This says that $n_0 > n_1 = n_2$.

5. There are 6 members on the Math Competition Committee. The problems are kept in a safe. There are ℓ locks on the safe and there are k keys, several for each lock. The safe does not open unless all of the locks are unlocked, and each key works on exactly one lock. The keys should be distributed to the 6 members of the committee so that each group of 4 members has enough keys to open all of the ℓ locks. However, no group of 3 members should be able to open all of the ℓ locks.
(a) Show that this is possible with $\ell = 20$ locks and $k = 60$ keys. That is, it is possible to use 20 locks and to choose and distribute 60 keys in such a way that every group of 4 can open the safe, but no group of 3 can open the safe.
(b) Show that we always must have $\ell \geq 20$ and $k \geq 60$.

Solution: There are $\binom{6}{3} = 20$ subsets of three members. For each such subset, there must be a lock that cannot be opened by the members of that subset. If two different 3-member subsets have the same lock that they cannot open, then there is a 4-member set, chosen from the union of these two 3-member subsets, that cannot open this lock. Since every 4-member subset should be able to open the lock, this is a contradiction. Therefore, there must be at least 20 locks. If some lock has only 2 keys, then there are 4 members who did not receive a key to this lock, so this 4-member subset cannot open the lock. Therefore, there must be at least 3 keys per lock, so there must be at least 60 keys. This proves (b).

For (a), assign one of the 20 locks to each 3-member subset and give a key for this lock to each of the 3 members of the subset. Then no 3-member subset will be able to open the lock assigned to the complementary 3-member subset. However, every 4-member subset intersects every 3-member subset, so every 4-member subset will have a key to every lock.