27^{th} ANNUAL (2005) UNIVERSITY OF MARYLAND
HIGH SCHOOL MATHEMATICS COMPETITION
PART II SOLUTIONS

1. The three little pigs are learning about fractions. They particularly like the number $x = 1/5$, because when they add the denominator to the numerator, add the denominator to the denominator, and form a new fraction, they obtain $6/10$, which equals $3x$ (so each little pig can have his own x). The 101 Dalmatians hear about this and want their own fraction. Your job is to help them.

   (a) Find a fraction y such that when the denominator is added to the numerator and also added to the denominator, the result is $101y$.

   (b) Prove that the fraction y (put into lowest terms) in part (a) is the only fraction in lowest terms with this property.

   Solution: Write $y = a/b$. We want $(a + b)/(b + b) = 101a/b$. This simplifies to $b = 201a$. Therefore, $a/b = 1/201$ is the only possibility, and it works.

2. A small kingdom consists of five square miles. The king, who is not very good at math, wants to divide the kingdom among his 9 sons. He tells each son to mark out a region of 1 square mile. Prove that there are two sons whose regions overlap by at least 1/9 square mile.

   Solution: One solution: There are 9 regions inside 5 square miles, so there must be at least 4 square miles of overlaps. There are $(9 \times 8)/(2 \times 1) = 36$ pairs of regions, so some overlap for a pair of regions must be at least $4/36 = 1/9$ square miles.

   Another solution: Suppose the sons try to choose their regions with less than 1/9 overlap. The first one uses up 1 square mile. The second adds more than 8/9, allowing for less than 1/9 overlap with the first. The third adds more than 7/9 new area, since he can have at most $1/9 + 1/9$ overlap with the first two. Continuing in this way, we find that after the 9th son has chosen, the area they have used is more than $1 + 8/9 + 7/9 + \cdots + 1/9 = 5$. Since the area of the kingdom is 5, this is impossible. Therefore, some overlap must have been at least 1/9.

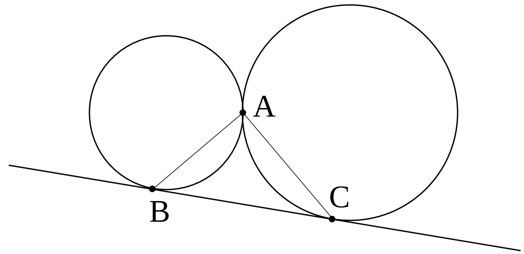
3. Let $\pi(n)$ be the number of primes less than or equal to n. Sometimes n is a multiple of $\pi(n)$. It is known that $\pi(4) = 2$ (because of the two primes 2, 3) and $\pi(64540) = 6454$. Show that there exists an integer n, with $4 < n < 64540$, such that $\pi(n) = n/8$.

   Solution: Note that $\pi(4)/4 > 1/8 > \pi(64540)/64540$. Therefore, there must be at least one n with $\pi(n)/n \geq 1/8 > \pi(n + 1)/(n + 1)$. If $\pi(n)/n$ is never 1/8, then the first inequality is strict, and it yields $8\pi(n) > n$. The second inequality yields $n + 1 > 8\pi(n + 1) \geq 8\pi(n)$. Combining our two results, we find that

   $$n + 1 > 8\pi(n) > n.$$

   Since n and $n + 1$ are consecutive integers and $8\pi(n)$ is an integer, this is impossible. Therefore, we must have had $\pi(n)/n = 1/8$. (*Remark:* A value of n that works is $n = 8472$. *Another remark:* Since the function $\pi(x)/x$ is not continuous, the intermediate value theorem does not apply. For example, $\pi(n)/n$ is never equal to 23/101.)

4. Two circles of radii R and r are externally tangent at a point A. Their common external tangent is tangent to the circles at B and C. Calculate the lengths of the sides of triangle ABC in terms of R and r.



Solution: Let x be the length of BC. Without loss of generality, we may assume that $R \geq r$. Let X be the center of the circle of radius r that is tangent to BC at B and let Y be the center of the circle of radius R tangent at C. The length of XY is $r + R$. Let Z be the point on YC that is a distance r from C. Since XB and YC are perpendicular to BC, they are parallel, which impies that XZ is parallel to and has the same length x as BC. Therefore, XZY is a right triangle with sides $x, R - r, R + r$. The Pythagorean theorem yields $x^2 + (R - r)^2 = (R + r)$, so $x^2 = 4rR$. Therefore, BC has length $2\sqrt{rR}$.

Draw a line from A perpendicular to BC and let it intersect BC at P and intersect XZ at Q. Triangle AQX is similar to triangle YZX. Let XQ have length y and AQ have length h. Since XY has length $R + r$ and YZ has length $R - r$, we have $h/r = (R - r)/(R + r)$. Also, from above, XZ has length $2\sqrt{rR}$, so $y/r = 2\sqrt{rR}/(R + r)$. By the Pythagorean theorem,

$$AB^2 = y^2 + (h + r)^2 = r^2 \left(\frac{4rR + 4R^2}{(R + r)^2} \right) = \frac{4Rr^2}{R + r}.$$

Therefore, AB has length $2r\sqrt{R/(R + r)}$.

Similar reasoning shows that the length of AC is $2R\sqrt{r/(R + r)}$.

5. There are 2005 people at a meeting. At the end of the meeting, each person who has shaken hands with at most 10 people is given a red T-shirt with the message "I am unfriendly." Then each person who has shaken hands only with people who received red T-shirts is given a blue T-shirt with the message "All of my friends are unfriendly." (Some lucky people might get both red and blue T-shirts, for example, those who shook no one's hand.) Prove that the number of people who received blue T-shirts is less than or equal to the number of people who received red T-shirts.

Solution: Let B be the number of pure blue people (that is, people who received a blue shirt but not a red shirt) and let R be the number of pure red people. The number of handshakes by pure red people is at most $10R$. Each pure blue person has shaken hands at least 11 times, since otherwise such a person would also be red.

We claim that each handshake by a pure blue person has been with a pure red person. This is a key point. If a pure blue person P had shaken hands with a blue-red person Q, then Q would have shaken hands with the non-red person P and therefore would not have received a blue shirt.

Therefore, the number of handshakes by pure blues is less than or equal to the number of handshakes by pure reds. This implies that $11B \leq 10R$, so $B \leq R$ (and it they are equal, then $B = R = 0$). Now add in the people who received both red and blue shirts. This increases the count for reds and the count for blues by the same amount. Therefore, the number of blues is less than or equal to the number of reds.