# THE 42$^{\text{nd}}$ ANNUAL (2021) UNIVERSITY OF MARYLAND HIGH SCHOOL MATHEMATICS COMPETITION

## PART I   SOLUTIONS

1.
$$\frac{\left(\frac{1/2}{3/4}\right)}{\left(\frac{5/6}{7/8}\right)} = \frac{\frac{1}{2}\cdot\frac{4}{3}}{\frac{5}{6}\cdot\frac{8}{7}} = \frac{2}{3}\cdot\frac{21}{20} = \frac{7}{10}.$$

The answer is **d**.

2. Jarad has walked $2774 \times 5280 =$ feet. This means

$$s = \frac{2774 \times 5280}{2.5} = 2774 \times 2112 = 5,858,688.$$

The answer is **a**.

3. The areas of each of the pizzas are as follows:

a. Huey's pizza is $10 \times 10 = 100$ square inches.

b. Dewey's pizza is $\pi 6^2 = 36\pi \approx 113$ square inches.

c. Louie's pizza has area $6 \times 24 = 144$ square inches.

The answer is **a**.

4. The number of minutes that Peter needs to run to finish the race is

$$\frac{10}{15} \cdot 60 = 40.$$

Since Peter sleeps for 4.5 hours, it will take him 5 hours and 10 minutes to finish the race. On the other hand, it takes Terrapin $10/2 = 5$ hours to finish the race. Therefore, the answer is **c**.

5. The dog crosses the $x$-axis precisely when $\cos(2\pi x) = 0$. This happens precisely twice in each of the intervals

$$[0,1), [1,2), \ldots, [19,20].$$

Thus there are 40 places where the dog crosses the $x$-axis. The answer is **d**.

6. Each even year the account balance is multiplied by 1.20, and each odd year the account balance is multiplied by 0.83. Therefore, after 100 years the account balance is

$$100 \times (1.2 \times 0.83)^{50} = 100 \times (0.996)^{50} < 100.$$

The answer is **a**.

7. The number of possible tags is $26^2 = 676$. Since, $2021 > 2 \times 676$ at least three people must have identical name tags. Since $2021 < 3 \times 676$ it is possible that no four people have identical name tags. The answer is **b**.

8. Each of the terms in brackets is of the following form:

$$\frac{1}{n}+\frac{2}{n}+\cdots+\frac{n-1}{n}=\frac{1+2+\cdots+(n-1)}{n}=\frac{n(n-1)/2}{n}=\frac{n-1}{2}.$$

The above sum is obtained using the arithmetic sequence sum. Since $n$ ranges from 2 to 100 we obtain the following sum:

$$\frac{1}{2}+\frac{2}{2}+\cdots+\frac{99}{2}=\frac{99\times 100/2}{2}=99\times 25=2475.$$

The answer is **d**.

9. The number of ways to rearrange 8 pieces is 8!. Swapping each identical pair of rooks, bishops and knights does not change the arrangement. Thus, the answer is

$$\frac{8!}{2\times 2\times 2}=7!$$

The answer is **c**.

10. Since the diagonal of $ABCD$ has the same length as the side of $M(ABCD)$, its area is double the area of $ABCD$. Therefore, the area of $M^{(n)}(ABCD)$ is $2^n$. The smallest $n$ for which $2^n \geq 2021$ is $n = 11$. The answer is **a**.

11. The expression can be written as follows:

$$n!+(n+1)!+(n+2)!=n!(1+(n+1)+(n+1)(n+2))=n!(n+2+(n+1)(n+2))=n!(n+2)^2.$$

We note that if $n \geq 10$, then $n!$ is divisible by 100, since $2, 5$, and 10 appear in the product $1\times 2\times\cdots\times n$. Therefore, we need to have $n < 10$.

For $n = 1, 2, 4$ the product $n!(n+2)^2$ has no factors of 5.

For $n = 3$, the product $3!\times 5^2$ is not divisible by 4.

For $n = 5, 6, 7, 9$ the product $n!(n+2)^2$ has only one factor of 5.

For $n = 8$, we get $8!\times 10$ which is divisible by 100. Thus, the desired positive integers are $n = 1, 2, 3, 4, 5, 6, 7, 9$. The answer is **d**.

12. The given equation is equivalent to

$$2(b^3+2b^2+3b+4)=2b^3+5b^2+b+2 \Rightarrow b^2-5b-6=0 \Rightarrow b=6,-1.$$

The answer is **a**.

13. Taking logarithm of both sides we obtain the following:

$$(\log x)^2=25 \Rightarrow \log x=\pm 5 \Rightarrow x=10^5, 10^{-5}.$$

The answer is **e**.

14. The given information means

$$(x+80200)(x-80200)=p^3,$$

for some prime $p$. Since $p$ is prime, we have two possibilities:

a. $x+80200=p^2$ and $x-80200=p$; or
b. $x+80200=p^3$ and $x-80200=1$.

Subtracting we obtain

$$2\times 80200=p^2-p \text{ or } p^3-1.$$

We see $2\times 80200=400\times 401$, and since 401 is prime, $p=401$ works. The second possibility does not work. So, $x=80200+p=80601$. The answer is **b**.

15. By completing the squares we notice that the second curve is also a circle:

$$(x-2)^2+(y-3)^2=1.$$

The distance between the centers of these two circles is $\sqrt{4+9}=\sqrt{13}$. Thus, the shortest distance $|PQ|$ is $\sqrt{13}-2$. The answer is **c**.

16. Since both quadratics have distinct real roots we must have: $a^2>4b$, and $b^2>4a$. Combining the two we obtain

$$\frac{a^2}{4}>b>2\sqrt{a}\Rightarrow a^2>8\sqrt{a}\Rightarrow a^4>64a\Rightarrow a^3>64\Rightarrow a>4.$$

Furthermore, for every such $a$, the inequality $a^2/4>2\sqrt{a}$ holds, and thus there is a value of $b$ that satisfies $a^2/4>b>2\sqrt{a}$. This means $a>4$ is the best inequality that describes all possible values of $a$. The answer is **e**.

17. $K$ be the foot of the altitude from $B$ to $AC$. We see that $|AH|=|BC|$ and $\angle KAH=\angle CBK$, since both are complementary to either $\angle ACB$ or $\angle AHB$, depending of whether $ABC$ is an acute or an obtuse triangle. Therefore, the triangle $AHK$ and $BCK$ are congruent. This implies $|AK|=|BK|$. Therefore, $\angle BAC=45°$ if it is acute and $135°$ if it is obtuse. The answer is **e**.

18. We have the following:

$$3^{15}+3^{11}+3^6+1=(3^5)^3+3\cdot(3^5)^2+3\cdot 3^5+1=(3^5+1)^3=244^3=4^3\cdot 61^3.$$

Therefore, the largest prime factor of this number is 61. The answer is **b**.

19. The first few terms of the sequence are

$$P_1=9999, P_2=9998, P_3=9995, P_4=9990, P_5=9983.$$

We guess that $P_k=9999-(k-1)^2$. Note that if this equality holds for $P_{k-1}$ and $P_{k-2}$, then by the given recursion we obtain

$$\begin{aligned} P_k &= 2(9999-(k-2)^2)-(9999-(k-3)^2)-2 \\ &= 9999-2(k-2)^2+(k-3)^2-2 \\ &= 9999-2k^2+8k-8+k^2-6k+9-2 \\ &= 9999-k^2+2k-1 \\ &= 9999-(k-1)^2 \end{aligned}$$

Therefore, $P_k$ is positive precisely when $9999>(k-1)^2$, i.e. $100>k-1$. This happens when $k=1,2,\ldots,100$. The answer is **e**.

20. Using the Pythagorean Theorem in triangle $AOM, BON$, and $COP$ we obtain the following:

$$|OA|^2 = |OM|^2 + |AM|^2$$
$$|OB|^2 = |ON|^2 + |BN|^2$$
$$|OC|^2 = |OP|^2 + |CP|^2$$

This implies

$$|AM|^2 + |BN|^2 + |CP|^2 = |OA|^2 + |OB|^2 + |OC|^2 - |OM|^2 - |ON|^2 - |OP|^2.$$

Similar argument shows this sum is also equal to $|BM|^2 + |CN|^2 + |AP|^2$. Therefore,

$$|AM|^2 + |BN|^2 + |CP|^2 = |BM|^2 + |CN|^2 + |AP|^2.$$

Substituting the given values we obtain:

$$9 + 16 + 16 = 25 + 4 + |AP|^2 \Rightarrow |AP|^2 = 12 \Rightarrow |AP| = \sqrt{12} = 2\sqrt{3}.$$

The answer is **c**.

21. Consider two points $A, C$ with $|AC| = 10$ and a ray $\overrightarrow{CX}$ for which $\angle ACX = 30°$. To create a triangle $ABC$ with given conditions, we need to draw a circle centered at $A$ of radius $c$. For this circle to intersect $\overrightarrow{CX}$ at least twice, the number $c$ must be more than the distance from $A$ to $\overrightarrow{CX}$. Dropping a perpendicular from $A$ to $\overrightarrow{CX}$ we get a $30 - 60 - 90$ triangle. Thus the distance from $A$ to $\overrightarrow{CX}$ is 5. Thus, we must have $5 < c$. For this circle to intersect the ray $\overrightarrow{CX}$ at two points we must also have $c < 10$. Therefore, $5 < c < 10$. The answer is **a**.

22. Let $m = n + 2$ and rewrite $(n + 15)^2$ as $(m + 13)^2 = m^2 + 26m + 13^2$. This is divisible by $m$ precisely when $m$ divides $13^2$. Since $m = n + 2$ is at least 3 we need to have $n + 2 = 13$, or 169. Thus, the possible values of $n$ are 11 and 167. Their sum is 178. The answer is **c**.

23. Since we are only interested in odd coefficients, we will replace each odd coefficient by 1 and each even coefficient by zero. In other words, we will consider all coefficients mod 2. This gives us the following:

$$(1 + x + x^2)^2 = 1 + 2x + 3x^2 + 2x^3 + x^4 = 1 + x^2 + x^4 \pmod 2 \qquad (*)$$

Repeating this process we have

$$(1 + x + x^2)^4 = 1 + x^4 + x^8 \pmod 2.$$

And hence

$$(1 + x + x^2)^{100} = (1 + x^4 + x^8)^{25} \pmod 2.$$

For simplicity we will set $y = x^4$ and consider $(1 + y + y^2)^{25}$. Writing 25 as a sum of powers of 2 and repeating $(*)$ we obtain the following:

$$\begin{aligned}(1 + y + y^2)^{25} &= (1 + y + y^2)^{16}(1 + y + y^2)^8(1 + y + y^2)\\ &= (1 + y^{16} + y^{32})(1 + y^8 + y^{16})(1 + y + y^2)\\ &= (1 + y^8 + y^{24} + y^{40} + y^{48})(1 + y + y^2) \pmod 2\end{aligned}$$

This yields 15 terms with exponents

$$0, 8, 24, 40, 48, 1, 9, 25, 41, 49, 2, 10, 26, 42, 50.$$

Thus, there are 15 terms with odd coefficients. The answer is **a**.

24. To each subset $S$ of $\{0, 1, \ldots, 19\}$ associate a sequence $a_0, a_1, \ldots, a_{19}$ for which

$$a_n = \begin{cases} 1 & \text{if } n \in S \\ 0 & \text{if } n \notin S \end{cases}$$

The size of $S \cap \{i, \ldots, i+9\}$ is $a_i + a_{i+1} + \cdots + a_{i+9}$. Suppose $0 \le i \le j \le 19$. The difference between the sizes of $S \cap \{i, \ldots, i+9\}$ and $S \cap \{j, \ldots, j+9\}$ is

$$\begin{aligned} a_i + a_{i+1} + \cdots + a_{i+9} - a_j - a_{j+1} - \cdots - a_{j+9} &= a_i + \cdots + a_{j-1} - a_{i+10} - \cdots - a_{j+9} \\ &= (a_i - a_{i+10}) + \cdots + (a_{j-1} - a_{j+9}) \end{aligned}$$

In order for $S$ to satisfy the given condition (c) we need the above sum to be $0, 1$, or $-1$. If we set $s_i = a_i - a_{i+10}$ for every $0 \le i \le 9$, we see that $s_i = 0, \pm 1$ for all $i$ and the given conditions can be summarized as follows:

a. $s_0 = 1$.

b. For every $0 \le i < j \le 9$ the sum $s_i + \cdots + s_{j-1}$ is $0, 1$ or $-1$.

Every $s_i$ that is zero does not change any of the sums in (b), above. Therefore, we can only focus on the non-zero $s_i$'s. Since $s_0 = 1$, for the sums to lie between $-1$ and $1$ we need the non-zero $s_i$'s to alternate: $1, -1, 1, \ldots$. Therefore, if $s_i$ is not zero, then it has exactly one possible nonzero value. If $s_i = 0$ then we have two possibilities: $a_i, a_{i+10} \in S$ or $a_i, a_{i+10} \notin S$. If $s_i = 1$, then we have one possibility $a_i \in S$ and $a_{i+10} \notin S$. If $s_i = -1$, then we have one possibility $a_i \notin S$ and $a_{i+10} \in S$. This means, for each pair $(a_i, a_{i+10})$ with $1 \le i \le 9$ there are three possibilities. So, there are $3^9$ possible subsets $S$. The answer is **b**.

25. We will show for each $1 \le i, j \le 20$ with $OA_i \perp OA_j$, we have

$$\frac{1}{|OA_i|^2} + \frac{1}{|OA_j|^2} = \frac{7}{10} \qquad (*)$$

Let $m$ be the slope of $OA_i$. The point $A_i$ then is of the form $(x, mx)$. Since $A_i$ is on the given ellipse, we must have:

$$\frac{x^2}{2} + \frac{m^2 x^2}{5} = 1.$$

This implies

$$x^2 = \frac{10}{5 + 2m^2}.$$

Therefore,

$$\frac{1}{|OA_i|^2} = \frac{1}{x^2 + m^2 x^2} = \frac{1}{x^2} \cdot \frac{1}{1 + m^2} = \frac{5 + 2m^2}{10(1 + m^2)}.$$

Similarly, since the slope of $OA_j$ is $-1/m$ we have

$$\frac{1}{|OA_j|^2} = \frac{5 + 2/m^2}{10(1 + 1/m^2)} = \frac{5m^2 + 2}{10(m^2 + 1)}.$$

Adding the two we obtain:

$$\frac{1}{|OA_i|^2} + \frac{1}{|OA_j|^2} = \frac{7 + 7m^2}{10(1 + m^2)} = \frac{7}{10}.$$

Since we may pair up $A_1, \ldots, A_{20}$ into ten pairs of points $A_i, A_j$ for which $OA_i$ and $OA_j$ are perpendicular, the answer is $10 \cdot \frac{7}{10} = 7$. The answer is **e**.

*Remark.* Note that when $m = 0$, the above argument is not valid, since $1/m$ is undefined. However, in that case $|OA_i|^2 = 1/2$ and $|OA_j|^2 = 1/5$ and thus the equality $(*)$ still holds.