25^{th} ANNUAL UNIVERSITY OF MARYLAND
HIGH SCHOOL MATHEMATICS COMPETITION
PART II
December 3, 2003, 1:00–3:00

1. (a) (15 points) Find three positive integers a, b, c whose sum is 407, and whose product (when written in base 10) ends in six 0's.

   (b) (15 points) Prove that there do NOT exist positive integers a, b, c whose sum is 407 and whose product ends in seven 0's.

2. (30 points) Three circles, each of radius r, are placed on a plane so that the center of each circle lies on a point of intersection of the other two circles. The region R consists of all points inside or on at least one of these three circles. Find the area of R.

3. (30 points) Let $f_1(x) = a_1x^2+b_1x+c_1$, $f_2(x) = a_2x^2+b_2x+c_2$ and $f_3(x) = a_3x^2+b_3x+c_3$ be the equations of three parabolas such that $a_1 > a_2 > a_3$. Prove that if each pair of parabolas intersects in exactly one point, then all three parabolas intersect in a common point.

4. Gigafirm is a large corporation with many employees.

   (a) (10 points) Show that the number of employees with an odd number of acquaintances is even.

   (b) (20 points) Suppose that each employee with an even number of acquaintances sends a letter to each of these acquaintances. Each employee with an odd number of acquaintances sends a letter to each non-acquaintance. So far, Leslie has received 99 letters. Prove that Leslie will receive at least one more letter.

   (*Notes:* "acquaintance" and "non-acquaintance" refer to employees of Gigafirm. If A is acquainted with B, then B is acquainted with A. However, no one is acquainted with himself.)

5. (a) (5 points) Prove that for every positive integer N, if A is a subset of the numbers $\{1, 2, \ldots, N\}$ and A has size at least $2N/3 + 1$, then A contains a three-term arithmetic progression (i.e., there are positive integers a and b so that all three of the numbers a,$a + b$, and $a + 2b$ are elements of A).

   (b) (25 points) Show that if A is a subset of $\{1, 2, \ldots, 3500\}$ and A has size at least 2003, then A contains a three-term arithmetic progression.