1999 SOLUTIONS, PART I

Problem 1. Answer: 2019.

Problem 2. Answer: 2046.

Problem 3. Answer: 7.

Problem 4. Answer: 2.

Problem 5. Answer: 130.

Problem 6. Answer: 7.

Problem 7. Answer: bca. 2^{10}=1024, so 2^{1999} is roughly 10^{600}. 1999^2 is roughly $4\cdot10^6$. Observe that $10=\log_2 1024 < \log_2 1999 < \log_2 2048=11$. Hence the third number is between 10^{10} and 10^{11}.

Problem 8. Answer: 6. Clearly 1,3,7,9,21,63 divide any such N. These are the only divisors of 63.

Problem 9. Answer: 24. 20+17+11=48 counts each freshmen twice.

Problem 10. Answer: cow, goat, horse. Cow: $5\cdot6^2=180$. Horse: $3\cdot pi\cdot5^2=75pi > 225$. Goat: $(1/2)22^2 3^{1/2}/2$, which is between 180 and 225, since $1.5<3^{1/2}<1.8$.

Problem 11. Answer: 7. The last digits of 777^n form a periodic sequence: 7,9,3,1,7,9,3,1,...

Problem 12. Answer: 1 hour. A team of 2 Supermen, 2 Batmen and 2 Cinderellas will peel 3+4+5=12 buckets in an hour. Therefore a team of 1 Superman, 1 Batman and 1 Cinderella will peel 6 buckets in an hour. Hence Superman will peel 6-5=1 bucket.

Problem 13. Answer: ACB. Suppose 100 calls are made, 10 of them 1 minute long, 10 - 5 min long, 30 - 10 min long, 30 - 20 min long, 20 - 30 min long. Plan A: 99·100+20·10·5=$109. Plan B: 10+50+300+600+600=$156. Plan C: 0.8·156+25=$149.80.

Problem 14. Answer: 166. There are 6·5·4=120 words with no repeating letters; 3·5·3=45 words in which one letter appears twice (3 choices for the repeated letter, 5 choices for the unrepeated letter, 3 choices for the position of the unrepeated letter); 1 word EEE with a triple letter.

Problem 15. Answer: 3:04 pm. At 2 pm they are 2 miles away from the lost paddle. D takes them back in 2/3 hour. It takes them 2/5 hour more to get to the hat which is 2 miles down the stream. The speed of the river is irrelevant.

Problem 16. Answer: 2/3.

Problem 17. Answer: $10G < g! < 10^G$. Choose k so that $10^G=g^k=10^{100k}$, k=G/100. Thus $10^G=g^{G/100} > g^g > g!$ On the other hand, $10G=10^{g+1}$,g! (most factors are much bigger than 10).

Problem 18. Answer: 29. Can pay: 6,7; 12,13,14; 18,19,20,21; 24,25,26,27,28; 30,31,32,33,34,35 and everything after that.

Problem 19. Answer: -4. $1-7/x+8/x^2+2/x^3=2(1/x-1/r)(1/x-1/s)(1/x-1/t)$. So, 8=-2(1/r+1/s+1/t).

Problem 20. Answer: 1. (ii) implies that f(n) can be large; (iii) implies that f(k)=k for k < f(n), and hence for all k.

Problem 21. Answer: $10^{15}< M < 10^{20}$. Let M_k be the number of steps required to order a list of k numbers.
The (k+1)st number will not be reached by the computer before it orders the first k numbers. The largest number of steps
to put the (k+1)st number in its place is k+(k-1)+...+1=k(k+1)/2. Hence $M_{k+1}=M_k+k(k+1)/2$. So M is approximately 10^{18}.

Problem 22. Answer: yes,no,no. Assuming the usual checkered coloring, one cannot remove squares of the same color.

Problem 23. Answer: 2bc cosw/(b+c). Let |AD|=x. The area of ABC is (bc sin2w)/2. The areas of ADB and ADC (which add up to the area of ABC) are (cx sinw)/2 and (bx sinw)/2.

Problem 24. Answer: 12. P(x)=k(x+1)x(x-1)(x-2) for some k. P(-2)=k(-1)(-2)(-3)(-4). P(3)=k·4·3·2·1=P(-2)=12.

Problem 25. Answer: $1+2(2^{1/2})/(3^{1/2})$. The centers of the spheres form a regular tetrahedron with edge length 2. The distance from a vertex of a face to the center of that face is $2(3^{1/2})/3=2/(3^{1/2})$. The distance from the top vertex to the bottom face is $(4-4/3)^{1/2}=2(2^{1/2})/(3^{1/2})$. The height of the bottom face is 1.