2000 SOLUTIONS, PART I

Problem 1. The numbers are: 2^{81}, 2^{18}, 2^{24}, 2^{64}, 2^{48}, Answer: a.

Problem 2. The areas are proportional to 500, 3·196=588, 2·256=512. Answer: e.

Problem 3. 1/2·2/3·3/4·4/5=1/5. Answer: a.

Problem 4. If Walter catches up to Timothy at 2:00+x min, then Walter spends x-12 min and Timothy spends x min. From 3·(x-12)=x we get x=18. Answer: d.

Problem 5. All primes except 2 are odd so M is not divisible by 4 and so cannot end with 2 or more zeros. However M is 2 times 5 times other integers and so ends with a zero. Answer: b.

Problem 6. One sheep eats one field in 5 days. Answer: d.

Problem 7. The losing rates are 1·$1000, 1/2·$1000, 1/3·$1000. The combined rate is (1+1/2+1/3)·$1000=(11/6)·$1000. They lose $2000 in 12/11 of an hour. Answer: e.

Problem 8. The area of the court is 50·94·144 square inches. The area of the coin is at least $3.14 \cdot 1.04^2/4>3/4$. Therefore the number of coins is less than 50·100·150·4/3=1 million. On the other hand each coin fits into a 1.04 by 1.04 square. One can easily fit (50·12-2)(94·12-2)>598·1000 such squares into the court. Answer: d.

Problem 9. A+2A+3A=180°. Hence A=30° and C is a right angle. Answer: e.

Problem 10. Answer: a.

Problem 11. Since the coefficients are integers, the polynomial cannot have exactly one irrational root. Answer: d.

Problem 12. Clearly a<13 and a>6. The solutions are: 7,42; 8,24; 9,18; 10,15; 12,12. Answer: e.

Problem 13. The area is pi^2 and pi is between 3 and 4. Answer: d.

Problem 14. If there are no draws, the total number of points is 3·10·9/2=135. Each draw decreases the total by 1 point. Answer: e.

Problem 15. Originally the pulp in the berries weighs 2 pounds and is 1/10th of the total weight. After 1 week the pulp (2 pounds) is 1/5th of the total weight. Hence the weight is 10 pounds. Answer: b.

Problem 16. The upper right vertex has coordinates (x,y) with $y=15-x^2=2x$. Hence x=3. Answer: e.

Problem 17. 2+2+3+3=10 gives 36. Answer: c.

Problem 18. The angle of 1 radian is a little less than 60°, so its sin is a bit less than sin 60°. The angle of 2 radians is a little less than 120°, so its sin is greater than sin 120°=sin 60°. The angle of 3 radians is a little less than 180°, its sin is close to 0. Answer: e.

Problem 19. From similar triangles CD/AB=3=(distance from P to CD)/(distance from P to AB). Answer: c.

Problem 20. The number n(n+1) must be a multiple of 20. Therefore one of n and n+1 must be a multiple of 5 and one (maybe the same one) must be a multiple of 4. The numbers are: 4,15,19,20,24,35,39,40,44,55,59,60,64,75,79,80,84,95,99,100. Answer: b.

Problem 21. The product is 1. The first 2 students gave correct answers. Answer: c.

Problem 22. Note that 4·5=20. Since 6 is a digit, the base is greater than 6. The bases (greater than 6) for which 20 ends with a 6 are 7, 14, ... Since 3506 is greater than the product should be in base 10, the base is < 10. Answer: a.

Problem 23. Answer: b.

Problem 24. f(x,1)=f(1,x)=f(0,x)+x+1=2x+1; f(x,2)=f(2,x)=2x+1+x+1=3x+2. In general f(x,y)=(x+1)(y+1)-1. Answer: e.

Problem 25. Consider ANY unbroken 20 hour period and let dscs, dscn, dncs, dncn be the total times when "dog is asleep, cat is asleep", "dog is asleep, cat is not", etc. Claim: all four times are equal. Proof. Break the 20 hour interval into the first 10 hours and second 10 hours and attach digits 1 and 2 to break each of the 4 times into 2 corresponding parts: dscs=dscs1+dscs2, dscn=dscn1+dscn2, etc. The dog's pattern has period 10; so dscs1+dscn1=dscs2+dscn2, etc. The cat's pattern is periodic with period 4. Hence during the second 10 hour period the cat switches its pattern to the opposite of the first 10 hour period. Hence dscn2=dscs1, dscs2=dscn1, dncs2=dncn1, dncn2=dncs1. Note that the dog sleeps exactly a half of any unbroken 10 hour period. Hence dscs1+dscs2=dscs1+dscn1=5 and the claim is true. It follows that in any 24 hour unbroken interval both animals will be asleep for at least 5 hours. It is easy to arrange the 24 hour period so that the dog sleeps during the last 4 hours. Answer: b.