2001 SOLUTIONS, PART I

Problem 1. The numbers are: 2^{81}, 2^{18}, 2^{24}, 2^{64}, 2^{48}, respectively. Answer: a.

Problem 2. Dividing all three by $(9!)^{1/2}$ yield 10, $9\cdot 10^{1/2}$, and $10^{1/2}\cdot 11^{1/2}$, respectively. Answer: e.

Problem 3. When x<=-2, |x|=-x and |x+2|=-x-2, sp the equation holds. It fails in all other cases. Answer: a.

Problem 4. Since log(2)+log(3)=log(6), the expression is equal to $10^{\log(6)}=6$. Answer: b.

Problem 5. $43^2=1849$ is the only perfect square in the 1800's, so he was born in 1849-43=1806. Answer: a.

Problem 6. Every blab is a blib, and some of these are blubs, so (since there are some blabs) some blibs are blubs. Answer: c.

Problem 7. If D=# of dogs and C=# of cats, then D+24=2C and D=2(C-x). Thus, D+24=D+2x, so x=12. Answer: b.

Problem 8. 29=(27+36+25+28)/4. Note that 29 is also the mean of all 5 numbers. Answer: d.

Problem 9. Label the trapezoid ABCD, with AB of length 20 on the bottom and CD of length 10 on the top. To find the length of the diagonal AD, drop a perpendicular from D onto AB, and let X be the place where it intersects. Then DXB is a right triangle, so $DX^2=89-5^2$. Thus, DX=8. Since DXA is a right triangle, $AD^2=15^2+8^2$. Answer: c.

Problem 10. We count the cases when Romeo (R) is to the left of Juliet (J) and Caesar (C) is to the left of Brutus (B) and multiply by 4. Label the positions 1,2,3,4,5,6. If R is in position 1, then J is in 2. If in addition, C is in 3 then there are 4 possibilities; if C is in 4, then there are two possibilites and nothing else is possible. So there are 6 possibilities when R is in 1. When R is in 2, J is in 3. If C is in 1, then there are 6 possibilities, but if C is in 4, there are 2 cases, giving a total of 8 cases when R is in 2. When R is in 3, J is in 4. If C is in 1, then there are 4 cases, while if C is in 2, then there are 4 cases, giving a total of 8 cases. When R is in 4, J is in 5. If C is in 1, then B can be in two places, giving a total of 4 cases. If C is in 2, then B must be in 6, giving 2 cases, and if C is in 3, then B must also be in 6, giving 2 cases for 8 cases total when R is in 4. When R is in 5, J is in 6. If C is in 1, there are 4 cases; if C is in 2, there are 2 cases, giving 6 cases. Thus, there are 6+8+8+8+6=36 cases satisfying these addtional constraints, so there are $4\cdot 36=144$ cases altogether. Answer: d.

Problem 11. Multiplying by sin(x) and simplifying yields $\sin^2(x)=\cos(x)$. Let u=cos(x). Then, since $\sin^2(x)+\cos^2(x)=1$, $u=1-u^2$. Applying the quadratic formula and throwing out a root since |u|<=1 yields $u=(5^{1/2}-1)/2$. Answer: c.

Problem 12. The total amount of chocoalate in a bunny is proportional to the CUBE of its height. So, assuming that the amount of chocolate in a 1-inch bunny is k, the amount of chocolate consumed by F is $(5k)^3=125k^3$, by M is $10\cdot (2.5k)^3$, and by B is $100\cdot k^3$. Clearly, B< F and F< M since $2.5^3>12.5$. Answer: c.

Problem 13. The proportion 17:27 implies that we want 510 lbs of Copper and 810 lbs of Tin. If A and B represent the amounts of each alloy, we have two equations in two unknowns: 1/3 A+2/5B=510 and 2/3 A+3/5B=810. Thus A=270 and B=1050. Answer: e.

Problem 14. If S is the sum of the integers from k+1 to k+n, then S+100 is the sum of k+n+1 to k+2n. Each of the n terms in the latter sum is n more than the corresponding entry in the former sum. That is, $n^2=100$, so n=10. Answer: a.

Problem 15. Larry (L) and Curly (C) drive for 3 hours (75 miles). Then C got out, and L drives backward for 2 hours. L picks up Moe and they drive for 3 more hours. Total time=8 hours. Answer: d.

Problem 16. Suppose that the radius of the circle is r. Let s be the side length of the square inside the semicircle, and let t be the side length of the square inside the full circle. We want to calculate s^2/t^2. In the semicircle, there is a right triangle with sides s,s/2, and hypotenuse r. Thus, $r^2=5s^2/4$, while the full circle gives a right triangle with sides t/2,t/2 and hypotenuse r. So $r^2=t^2/2$. So $s^2/t^2=2/5$. Answer: c.

Problem 17. (0.1 mm/day)(1 km/10^6 mm)(1 mile/1.6 km)(1 day/24 hours)= $2.6\text{x}10^{-9}$ mph. Answer: e.

Problem 18. They start fighting at 1:00pm. At that time they have completed 1/3+1/4=7/12 of the job. Tom starts painting again at 1:10 and finishes in $(5/12)\cdot 3=5/4$ hours. Answer: b.

Problem 19. Let t be the side length of the triangle. There is a right triangle with sides h,t/2 and hypotenuse t. So $h^2=3t^2/4$. Thus, the area of the triangle is 1/2 $th=h^2/3^{1/2}$. The area of the square is s^2, so $h/s=3^{1/4}$. Answer: a.

Problem 20. $(4.5)^2+(1.5)^2=5/8(6^2)$. Answer: c.

Problem 21. Since $3^8 > 5000$, the only numbers are 1^8+1^8, 1^8+2^8, 1^8+3^8, 2^8+2^8, 2^8+3^8. Answer: a.

Problem 22. 2008! is the smallest factorial that is divisible by 2^{2001}. There are 1004 numbers <= 2008 that are divisible by 2. Of these, 502 are divisible by 4. Of these, 251 are divisible by 8. Of these 125 are divisible by 16. Of these 62 are divisible by 32. Of these, 31 are divisible by 64. Of these, 15 are divisible by 128. Of these, 7 are divisible by 256. Of these, 3 are divisible by 512 and finally 1 is divisible by 1024. Answer: d.

Problem 23. Any root of the polynomial will divide $420=1\cdot 4\cdot 3\cdot 5\cdot 7$. Thus, the largest number of distinct roots occurs with the polynomial $(x-1)^3(x-2)^2(x-3)(x-5)(x-7)$ or $(x-1)^4(x-4)(x-3)(x-5)(x-7)$. In either of the two cases, there are 5 distinct roots. Answer: b.

Problem 24. Huey can get any of 4 duck's hats, and each of these are symmetric. Assume that Huey gets Dewey's hat. If in addition Dewey gets H's hat, then there 2 possibilities for which hat Donald gets. If Dewey does NOT get H's hat, then there are 3 symmetric possibilities depending on whose hat he gets. Assume Dewey gets Donald's hat. Then there are 3 possibilities. Thus, by symmetry, there are $2+3\cdot 3=11$ arrangements that start with Huey getting Dewey's hat. By symmetry again, there are $4\cdot 11=44$ possibilities in all. Answer: c.

Problem 25. Let r be the distance from the center of the circle to a vertex and let s be the length of one side of the original octagon. Let T denote the isocoles triangle with two sides of length r and having base s. The area of the whole octagon is 8 times the area of this triangle. By the Law of Cosines, $s^2= r^2+ r^2- 2r^2\cos(360°/8)=r^2(1- 2^{1/2})$. Now let A be the midpoint of one side of the octagon, let B be the midpoint of an adjacent side, and let C be the center of the circle. Let D denote the vertex of the large octagon connecting the adjacent sides. Let x denote the length of BC. The area of the smaller octagon is 8 times the area of triangle ABC, so we are asked to find the ratio of the area of ABC to the area of T. Since the triangle ABC is similar to T, this ratio is equal to $(x/r)^2$. But CBD is a right triangle with sides x, s/2 and hypotenuse r. So $r^2=x^2+s^2/4$. Combining this with the equation above gives $x^2=r^2(2-2^{1/2})/4$. Answer: d.