2002 SOLUTIONS, PART I

Problem 1. By inspection z < 0, 0 < x < 1, and y > 1. Thus, z < x < y. Answer: e.

Problem 2. If S= my son's age now then 3S-5=4(S-5). Thus S=15. Answer: b.

Problem 3. The distance between 2x and 4 is at most 6, so the distance between x and 2 is at most 3. So -1 < = x < = 5. Answer: d.

Problem 4. The equation of the line containing the two points is y=2x-1. Answer: c.

Problem 5. Let S=amount Snow White earned. Then S+7· 20=8(S-3.5). So 7S=168, hence S=$24. Answer: c.

Problem 6. Setting the two equations equal and simplifying gives x^2-4x+3=0. So x=1 and 3, which implies y=2 and 0. Answer: a.

Problem 7. A=pi· a^2, B=pi· b^2, and C=pi· c^2. Since c^2=a^2 + b^2, C=A+B. Answer: a.

Problem 8. $\log_2(2)=1$ and $\log_3(9)=2$. Thus $\log_4(x)=3$, so $x=4^3$. Answer: e.

Problem 9. Since 7 Muppets eat 7 cookies in 7 minutes, each Muppet eats a cookie in 7 minutes. So each of the 14 Muppets can eat 2 cookies in 14 minutes, for a total of 28 cookies. Answer: c.

Problem 10. cos(-x)=x and $(\sin(x)+\cos(x))^2=\sin^2x+\cos^2x+2\sin(x)\cos(x)$, which is equal to 1+sin(2x). Answer: a.

Problem 11. The path of the bee forms a right triangle. Since he flew south for 1 hour and west for 3/4 hour, it takes him 5/4 hours to fly home (by the Pythagorean Theorem). In total, he spends 1+3/4+5/4=3 hours flying and he spends 1.5 hours in fields. Answer: b.

Problem 12. If r is the only solution, then b^2-4c=0. Thus, the quadratic formula yields r=-b/2, so b=-2r. From the first equation b/c=4/b, hence b/c=4/(-2r)=-2/r. Answer: b.

Problem 13. A regular 2002-gon is very nearly a circle. If a circle has circumference (i.e., perimeter) 1, then its radius is 1/(2pi). Thus, its area is $pi(1/(2pi))^2$=1/(4pi). Since pi is slightly larger than 3, the area is very nearly 1/12. Answer: d.

Problem 14. On each of the five days, the probability that he will eat a chocolate with a red wrapper is 3/5. Answer: e.

Problem 15. Let p be the cost of a pen. Then 1100< = 9p < = 1199, so 123 < = p < = 133. As well, 1500< = 13p < = 1599, so 115< = p < = 123. Thus, p=$1.23, so x=7 and y=99. Answer: d.

Problem 16. There are 9 1-digit palindromes (excluding 0), there are 9 2-digit palindromes (11 through 99), there are 9· 10=90 3-digit palindromes, and there are 10 4-digit palindromes that begin with 1 (the second digit can be anything). Combining this with 2002, you get 9+9+90+10+1=119 palindromes between 1 and 2002 (inclusive). Answer: e.

Problem 17. First, if any of x,y,z are zero, then it is easy to see that the other two must be 0 as well. Thus, (0,0,0) is one possible triple and every other triple has all 3 of x,y,z not equal to 0. If x,y,z are nonzero, then since xy=z and xz=y, x(xy)=y, hence x^2=1, so x=1 or -1. If x=1, then y=z and yz=x=1, so (1,1,1) and (1,-1,-1) are the only solutions. If x=-1, then y=-z and yz=-1, so (-1,1,-1) and (-1,-1,1) are the only solutions in this case. Thus, there are exactly 5 triples that satisfy the equations. Answer: d.

Problem 18. Each sparrow has 8 toes and one beak. Thus, in any group of sparrows, the difference between toes and beaks must be divisible by 7. In particular, there cannot be 80 more toes than beaks. Answer: c.

Problem 19. 95_b=9b+5=140. Thus, b=15. Answer: b.

Problem 20. During the course of travelling from the SW corner to the NE corner, you must travel 6 blocks North and 3 blocks East. Thus, you are travelling 9 blocks and you are free to choose which 3 times you travel East. So, the number of possible paths is (9 choose 3)=84. Answer: d.

Problem 21. Let W(n) be the maximal number of wires that can be stretched between n people equally spaced around a circle without any crossings. First, note that the requirement that the people be equally spaced is a red herring - W(n) is also equal to the number non-intersecting wires between n people located anywhere on the circle. We claim that W(n)=2n-3 for all n > = 2. Clearly, W(2)=1 and W(3)=3. Assume n > 3 and W(k)=2k-3 for all 2 < = k < n. We argue that W(n)=2n-3. First, suppose we have any maximal arrangement of wires between n-1 people. By assumption, such an arrangement would have 2(n-1)-3=2n-5 wires. If we add a new person between any two adjacent people, then we clearly can add two new wires: one between the new person and his left-hand and right-hand neighbors. Thus, W(n) is at least 2n-3. To see that we cannot do better, suppose we have any arrangement of wires between n people. If there are only wires between adjacent people, then there would be only n wires, which is less than 2n-3, since n > 3. So assume that there is a wire between two non-adjacent people, call them A and B. This wire splits the n people into two groups of people: One has size 2 < = k < n, while the other has size n-k+2 (A and B are included in both groups). By assumption, the number of wires in the first group is at most W(k)=2k-3, while the number of wires in the second group is at most W(n-k+2)=2(n-k+2)-3=2n-2k+1. Hence, the total number of wires in the original arrangement is at most W(k)+W(n-k+2)-1, which is at most (2k-3)+(2n-2k+1)-1=2n-3. [The -1 term is present since the wire between A and B is included in both subarrangements.] Answer: c.

Problem 22. Let r=speed of the rabbit, let c=speed of the train, t=total elapsed time and let d=distance travelled by the rabbit. Clearly, d=rt and, since the train moves a total distance of 1 mile, t=1/c, so d=r/c and r=cd. As well, the total time 1/c is equal to 1/(r-c) [the time the rabbit is moving with the train] + 1/(r+c) [the time the rabbit is moving opposite the train]. Substituting cd for r in this equation yields 1/c=1/(cd-c)+1/(cd+c). Multiplying both sides by c gives 1=1/(d-1)+1/(d+1), so (d-1)(d+1)=2d. Thus, d=(1+$2^{1/2}$)/2. Answer: a.

Problem 23. Suppose a and c are integers and there is a right triangle with side lengths a and 17 and hypotenuse c. Then c^2-a^2=17^2. So (c-a)(c+a) is equal to a prime squared. Since a is not 0, it must be that c-a=1 and c+a=17^2=289. So a=144 and c=145 is the only possibility. Answer: e.

Problem 24. Let N=number of days until all 7 dwarfs visit. Then N is the smallest integer that is divisible by 1,2,...7. So N=3· 4· 5· 7=420. In these 420 days, Dwarf #1 visits 420 times, #2 visits 210 times, #3 visits 140 times, #4 visits 105 times, #5 visits 84 times, #6 visits 70 times, and #7 visits 60 times. Answer: e.

Problem 25. First, it is possible to have 16 participants (their scores could be 000,011,022,033,101,112,123,130,202,213,220,231,303,310,321,332). The general pattern is that the scores are of the form (a,b,c) where a and b are anything (i.e., 16 possibilities) and c=a+b (mod 4). So, for any two participants, if their scores are the same on one of the first two problems but are different on the other, then their score on the third problem will also be different. To see that more than 16 participants is not possible, suppose that there were at least 17 participants. Since there are only 4 possible scores on the first question, there would be a group of at least 5 participants that got the same score on the first question. But then, since there are only 4 answers for the second question, at least two of these 5 would have the same score on the second question. Thus, this pair would have the same score on at least two of the questions, contradiction. Answer: d.