# 2003 SOLUTIONS, PART I

*Problem 1.* The numbers are $2^{243}$,$2^{125}$, $3^{32}$,$3^{25}$, and $5^9$, respectively. The first is larger than the second, and since $8=2^3$, the first is equal to $8^{81}$ which clearly dominates the other numbers. Answer: a.

*Problem 2.* In any circle, the circumference is proportional to the radius (i.e., C=2pi· r), and the area is proportional to the square of the radius (i.e., A=pi· $r^2$). The first statement says that the radius of Batman's pizza is 1.2 times the radius of Robin's, hence the area of Batman's pizza is $(1.2)^2=1.44$ times the area of Robin's. Answer: e.

*Problem 3.* There are 12 large dogs, 5 of which are mutts, so 7 are pure breeds. Answer: b.

*Problem 4.* If x satisfies the inequality, then 3x is at least 13 units away from -4, hence 3x is either less than -17 or greater than 9. Answer: b.

*Problem 5.* If one considers all triangles with one side of length a and another of length b, and one varies that angle between these two sides, the length of the third side (i.e., the side opposite the angle increases). Since we know that $a^2+b^2=c^2$ whenever the angle is a right angle, it follows that the angle must be obtuse whenever $a^2+b^2 < c^2$. Note: The numerical relationship is given by the Law of Cosines, $a^2+b^2$ -2ab cos(theta)= $c^2$. Answer: b.

*Problem 6.* $\log_2(8)=3$ and $\log_3(9)=2$, so x should satisfy $\log_5(x)=3-2=1$. Since $5^1=5$, x=5. Answer: d.

*Problem 7.* In the smallest such class there would be exactly one boy. So, if n represents the size of such a class, then 1/n < 7%. Thus, we want the smallest n so that 1/n < 7/100, hence n=15. Answer: c.

*Problem 8.* Let Q be the number of questions on the exam. Since all the numbers are integers, Q is divisible by 2, 3, 4 and 5. 60 is the only number < 100 divisible by all of these. Now each of the sentences say that each Beatle got exactly 37 questions correct. Answer: e.

*Problem 9.* Let s denote the length of a side of the cube. Then the `bottom diagonal' has length $s \cdot 2^{1/2}$ and the main diagonal is the hypotenuse of a right triangle with edges s and $s \cdot 2^{1/2}$. By the Pythagorean theorem, the length of the main diagonal is $s \cdot 3^{1/2}$. But the volume of the cube is $s^3$. Setting these equal, cancelling out s, and taking a square root yields $s=3^{1/4}$. Answer: e.

*Problem 10.* All right triangles with one angle of size x are similar, so choose one where the side opposite x has length 1 and the side adjacent to x has length 3. Clearly, tan x = 1/3. But now the hypotenuse has length $10^{1/2}$, so sin x $=1/10^{1/2}$. Answer: d.

*Problem 11.* Let x=Jack's age now and y=Jill's age now. So r=x/y and we want to find the ratio y/(y-(x-y)), or y/(2y-x). Dividing numerator and denominator by y yields 1/(2-r). Answer: d.

*Problem 12.* Note that $25=10^{\log(25)}$, so
$25^{1/\log(25)}=(10^{\log(25)})^{1/\log(25)}=10^1=10$. Answer: c.

*Problem 13.* The statements give us two equations, $s^2$=pi· $r^2$ and 4s=2pi· r. The first implies s/r=$\text{pi}^{1/2}$, while the second implies s/r=pi/2. Taken together, these would imply pi=0 or 4, both of which are false. Hence Jim is wrong. Answer: e.

*Problem 14.* The parabola $y=x^2+bx+c$ intersects the line y=dx in exactly one point precisely when the equation $x^2+bx+c=dx$ has exactly one solution. The equation $x^2+(b-d)x+c=0$ has one solution when $(b-d)^2$- 4· 1· c=0. Since b,c,d are all integers, this implies d-b is even. Answer: b.

*Problem 15.* The first statement translates to (LF) implies NOT(GE), the second to (W) implies (LF) and the third (T) implies (W), where (LF) abbreviates "loves fish" (GE) abbreviates green eyes, (W) abbreviates "has whiskers" and (T) abbreviates "has a tail". Putting these together yields (T) implies NOT(GE), which is the translation of sentence (a). Answer: a.

*Problem 16.* For x > 1, the value of $y^y$ (when $y=x^x$) is strictly increasing. When x=4, $y=4^4=256$ and
$y^y=256^{256} < 10^{768} < 10^{2003}$ (since $256 < 10^3$). When x=5, y=3125 so $y^y > 10^{2003}$, so the value of x making $y^y=10^{2003}$ is between 4 and 5. Answer: c.

*Problem 17.* Let c=number of questions answered correctly and w=number wrong (answered incorrectly). Then 8c-5w=13 and c+w < = 20. The possibilities can be checked by hand, but we can restrict the search by looking at the first equation modulo 5 and 8. Modulo 5, we have 3c is congruent to 3 (mod 5), hence c is congruent to 1 (mod 5), hence c=1,6,11,or 16. Similarly, -5w is congruent to 5 (mod 8) hence w is congruent to -1 (mod 8), hence w=7 or 15. The only solution is c=6, w=7. Answer: c.

*Problem 18.* $(a+(1/a))^2=a^2+2+1/a^2=3^2=9$, so
$(a-(1/a))^2=a^2-2+1/a^2=9-4=5$. Answer: a.

*Problem 19.* Let ABC be any right triangle, where angle A is the right angle. Let D be the point on BC where the altitude from A intersects the segment BC and let h=the length of the altitude AD. Let x=the length of BD and y=length of DC. Then the length of $AB=(h^2+x^2)^{1/2}$ and the length of $AB=(h^2+y^2)^{1/2}$ Since ABC is similar to DAC, h/y=(length of AB)/(length of AC), so
$h^2/y^2=(h^2+x^2)/(h^2+y^2)$, hence $h^2=xy$. So the area of the triangle is given by $1/2(x+y)h=1/2(x+y)(xy)^{1/2}$. In this problem, just plug in x=25, y=2003. Answer: a.

*Problem 20.* Let A,B,C denote the part of the job done by each pig in one hour. Then we get three equations, namely 2A+2B=1, (6/5)A+(6/5)C=1, and (3/2)B+(3/2)C=1. Solving these yields A=1/3, B=1/6, C=1/2, so A+B+C=1. That is, working together it takes the three pigs exactly one hour to dig the moat. Answer: e.

*Problem 21.* Let D=distance between the ports, C=rate of current, and B=rate of the boat in still water. Then D/(B+C)=5, while D/(B-C)=7. Taking reciprocals yields C/D + B/D=1/5 and B/D-C/D=1/7. Subtracting the first equation from the second yields 2(C/D)=2/35, hence D/C=35. Answer: b.

*Problem 22.* Let $a_n$ be the number of shares held by the $n^{th}$ shareholder and let T denote the total number of shares held. Without loss assume $a_n$ < = $a_m$ whenever n < = m. Then $a_1+...a_{1100}$ > = T/2. If we are trying to maximize $a_{2003}$ without increasing T, we should have $a_t=a_{1100}$ for all t between 1101 and 2002. Also, to keep $a_{1100}$ as small as possible we should have $a_t=a_{1100}$ for all t between 1 and 1099. Thus, letting $a=a_{1100}$ and $c=a_{2003}$, we have 1100a=T/2 and 2002a+c=T. Hence c=198/2200 T, so c/T=9/100. Answer: a.

*Problem 23.* The number of 4-digit numbers whose digits are strictly decreasing is in 1-1 correspondence with the number of 4-element subsets of {0,1,...,9}. Thus, the number of such sequences is (10 choose 4)=10· 9· 8· 7/ (4· 3· 2· 1)=210. Answer: d.

*Problem 24.* There are 8 sequences of the form 0001aaa, 4 of the form 10001bb, 4 of the form c10001c, 4 of the form dd10001, and 8 of the form eee1000. This gives 28 sequences, but it double-counts the sequence 0001000, so there are 27 in all. Answer: b.

*Problem 25.* Any handshake will preserve the relative parity of the 3 animals. That is, after any number of moves, the number of Trolls (T's) and the number of Dragons (D's) will either both be even, or they will both be odd. As well, the number of Griffins (G's) will have the opposite parity. At the end of any play of the game, the number of G's cannot be zero, since it's parity always differs from the other two. So there are only G's left at the end. Furthermore, since there are an even number of T's and D's, there must be an odd number of G's at the end of any play of the game. It is easy to see that there are plays of the game that result in any odd number less than 4003 G's being left. For example, to see that 25 Griffins is possible, first have a G and a D shake hands, resulting in 2002 T's and D's and 2001 G's. Then, after 13 T+D handshakes there are 1989 T's and D's, and 2014 G's. Now do 1989 rounds of triple handshakes, first T+G, then D+G, then T+D. Answer: b.